Nilsson Model

- Spherical Shell Model
- Deformed Shell Model
  - Anisotropic Harmonic Oscillator
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Spherical shell model

Nuclear properties described in terms of nucleons considered as independent particles moving in an average potential create by all nucleons.

Experimental evidence for shell effects:

Existence of magic numbers: 2,8,20,28,50,82,126

- Large single particle separation energies
- Nuclei are strongly bound at shell closures

Derivation of the average field from microscopic two-body forces (selfconsistent Hartree-Fock method).

Assume the existence of such a potential and construct it phenomenologically

Characteristics of the potential:

$$\left(\frac{\partial V(r)}{\partial r}\right)_{r=0} = 0 \qquad \left(\frac{\partial V}{\partial r}\right)_{r<R_0} > 0 \qquad V(r) \simeq 0, \quad r > R_0$$

Spherical potentials

Infinite square well	Harmonic oscillator	Woods-Saxon potential
$V(r) = -V_0 \quad \text{for } r \le R$ $= +\infty \quad \text{for } r > R$	$V(r) = \frac{1}{2} M\omega_0^2 r^2$	$V(r) = -\frac{V_0}{1+\exp\left[(r-R)/a\right]}$

Eigen-functions

$\psi \sim j_\ell(kr) Y_{\ell m}(\Omega)$	$\psi \sim R_{n\ell}(r) Y_{\ell m}(\Omega)$ $R_{n\ell}(r) = r^\ell e^{-\frac{1}{2}r^2} L_{n-1}^{\ell+1/2}(r^2)$	numerically

Eigen-energies

$E(n,\ell) = \frac{\hbar^2}{2MR^2}\xi_{n\ell}^2$ $\xi_{n\ell}$: root of $j_\ell(\xi) = 0$	$E(n,\ell) = \hbar\omega_0(2n+\ell+3/2)$ $= \hbar\omega_0(N+3/2)$	intermediate

Spherical potentials

Woods-Saxon potential

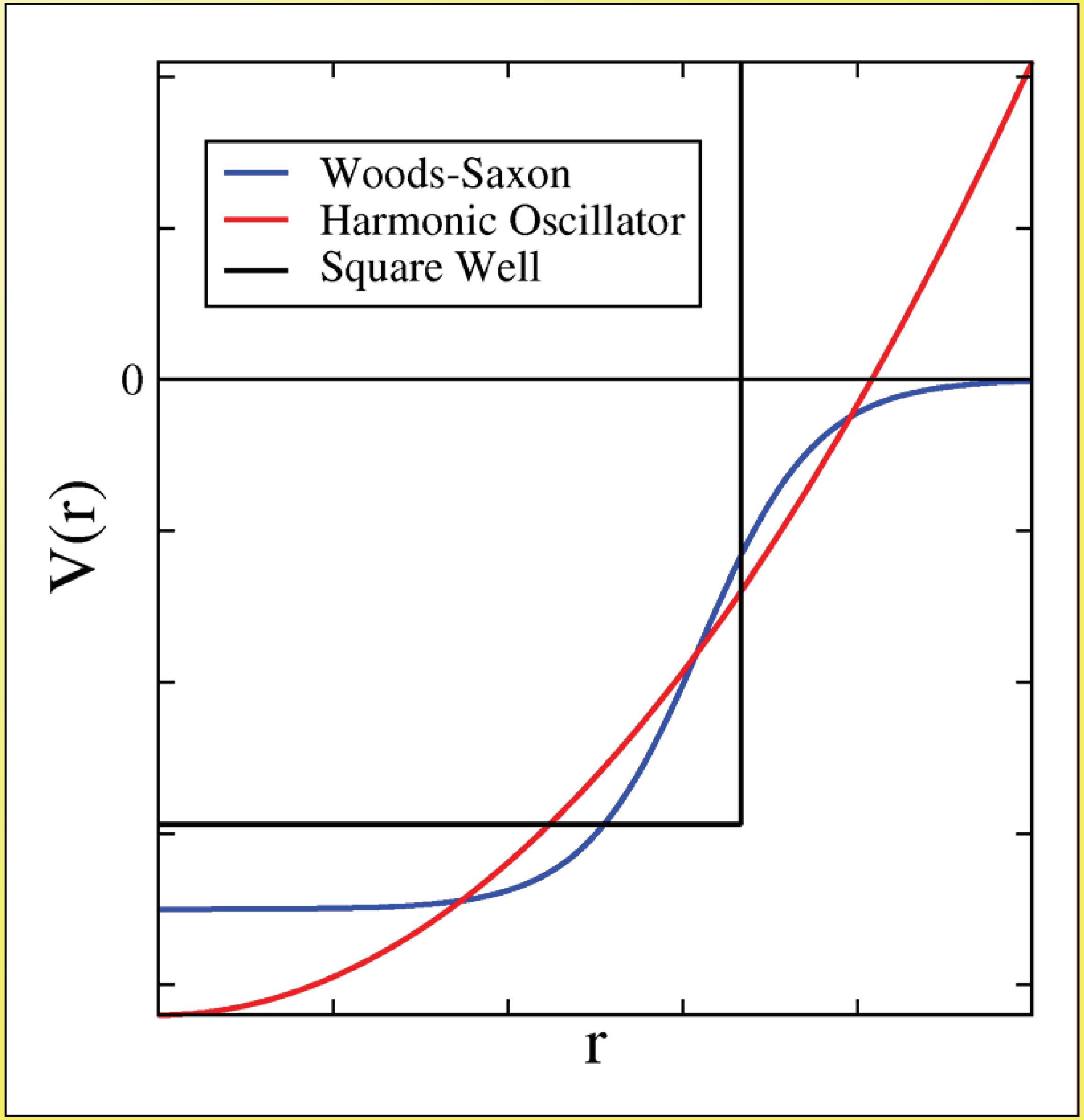
$$V(r) = -\frac{V_0}{1+\exp\left[(r-R)/a\right]}$$

Harmonic oscillator

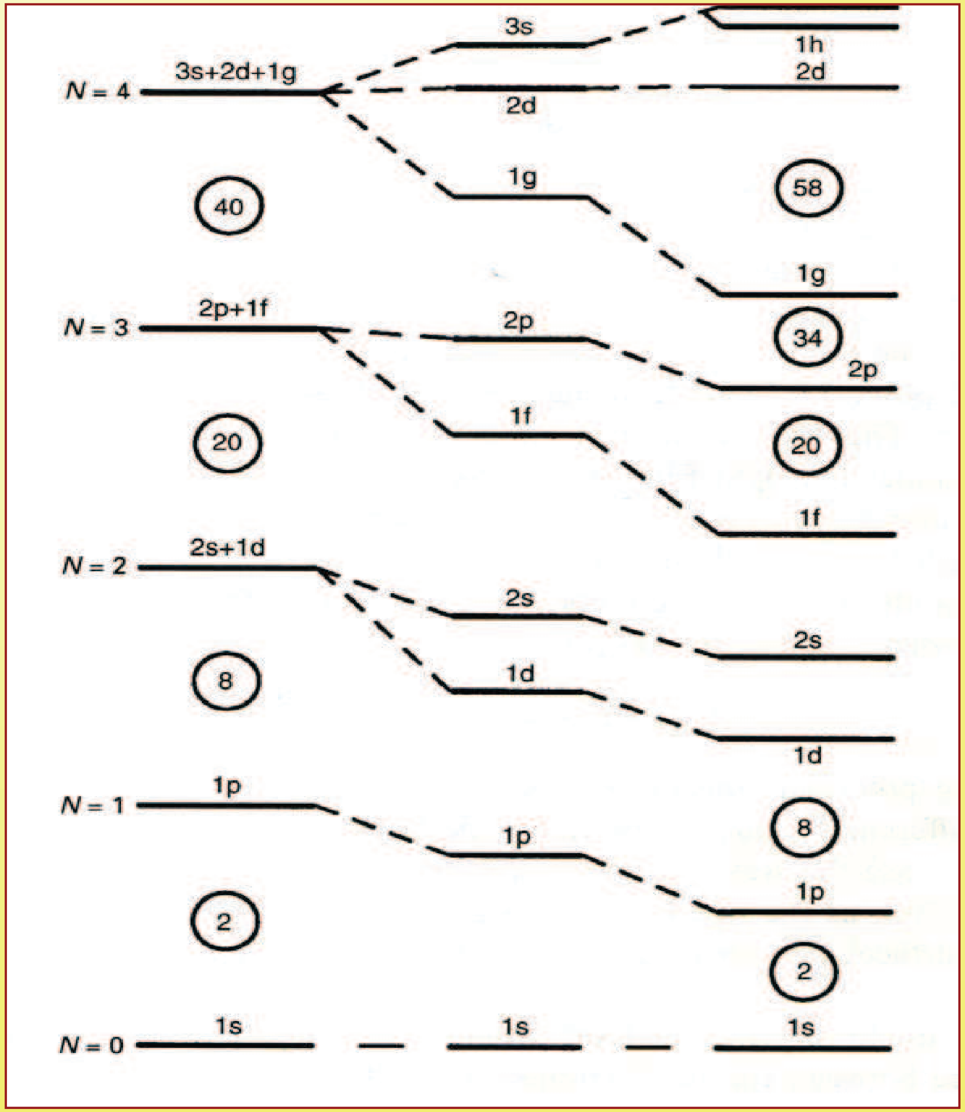
$$V(r) = \frac{1}{2} M \omega_0^{\,2} r^2$$

Infinite square well

$$\begin{aligned} V(r) &= -V_0 \quad \text{for } r \le R \\ &= +\infty \quad \text{for } r > R \end{aligned}$$



Spherical potentials



H.O. W.S. Square

Spherical potentials & spin-orbit

$$V(r)=\frac{1}{2}M\omega_0^{\,2}r^2+C\vec{\ell}\cdot\vec{s}+D\vec{\ell}^{\,2}$$

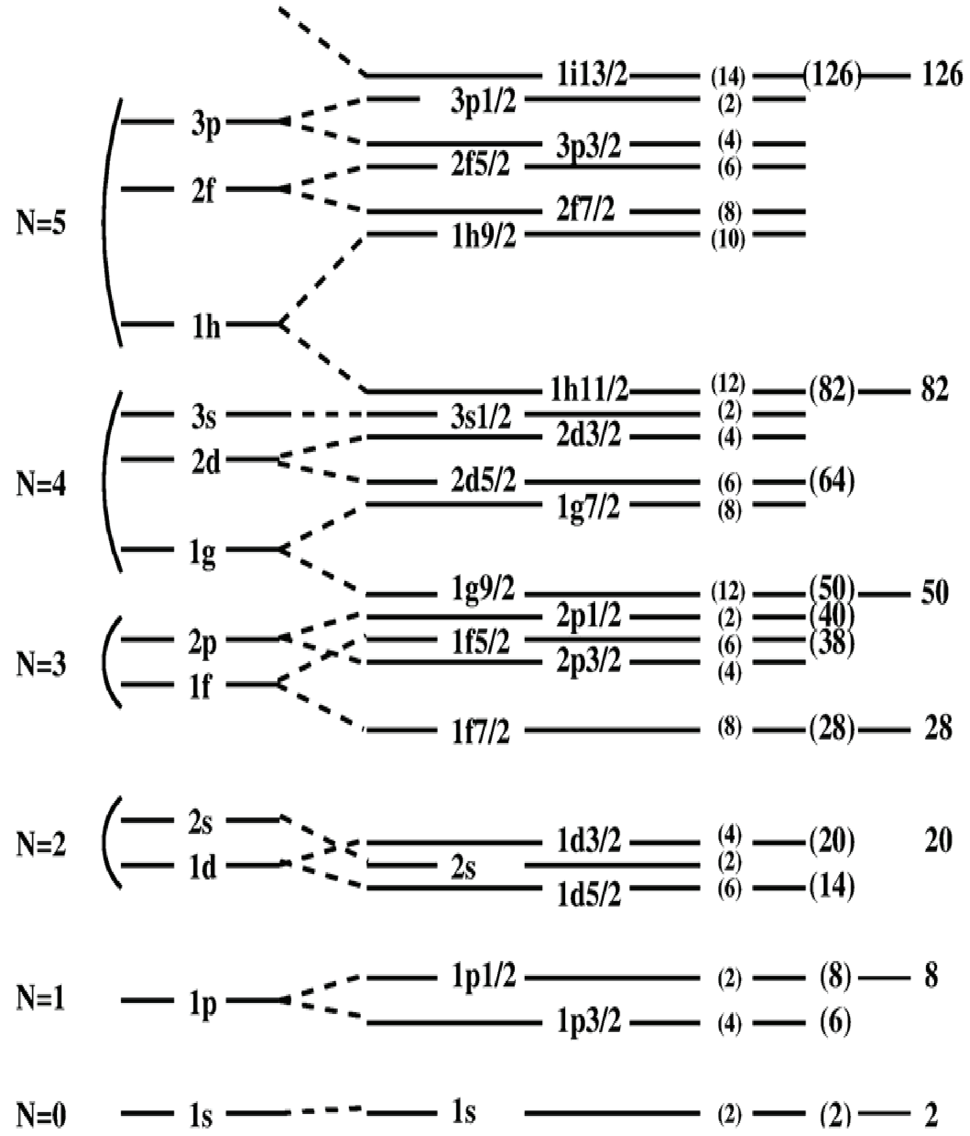
$$\vec{\ell}\cdot\vec{s}=\frac{1}{2}\left(\vec{j}^{\,2}-\vec{\ell}^{\,2}-\vec{s}^{\,2}\right)$$

$$E(n,\ell)=\hbar\omega_0\left(2n+\ell+3/2\right)+D\ell\left(\ell+1\right)+C\varepsilon_{SO}$$

$$\begin{cases}\varepsilon_{SO}=\ell & \text{for } j=\ell+\frac{1}{2}\\ \varepsilon_{SO}=-\ell-1 & \text{for } j=\ell-\frac{1}{2}\end{cases}$$

$$\Delta\varepsilon_{SO}=2\ell+1$$

Spherical mean-field



Deformed shell model

Spherical potential well valid for closed shells

Far from closed shells: deformed single particle potential

Experimental evidence:

- Existence of rotational bands: I(I+1) spectra
- Large quadrupole moments and quadrupole transition probabilities
- Single particle structure

Anisotropic Harmonic Oscillator

Generalized Woods-Saxon

$$V(r,\theta,\varphi) = -V_0\left[1+\exp\left(\frac{r-R(\theta,\varphi)}{a(\theta,\varphi)}\right)\right]^{-1}$$

$$V_{LS} = \lambda\left(\vec{\nabla}V(r,\theta,\varphi)\wedge\vec{p}\right)\cdot\vec{s}$$

Anisotropic Harmonic Oscillator

Ellipsoidal distribution: Anisotropic Harmonic Oscillator as average field

$$H_0 = -\frac{\hbar^2}{2m}\Delta + \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$$

Frequencies are proportional to the inverse of the ellipsoid axes $\quad \omega_i = \overset{0}{\omega}_0 \frac{R_0}{a_i}$

For axially symmetric shapes,

we introduce the parameter δ

$$\delta = \frac{\omega_\perp - \omega_z}{\omega_0}$$

$$\begin{cases} \omega_\perp^2 = \omega_x^2 = \omega_y^2 = \omega_0^2\left(1 + \frac{2}{3}\delta\right) \\ \omega_z^2 = \omega_0^2\left(1 - \frac{4}{3}\delta\right) \end{cases}$$

From volume conservation

$$\omega_x\omega_y\omega_z = \left(\overset{0}{\omega}_0\right)^3 \qquad \omega_0(\delta) = \overset{0}{\omega}_0\left[1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3\right]^{-1/6}, \qquad \delta \simeq 0.95\beta$$

Anisotropic Harmonic Oscillator

Introducing dimensionless coordinates through the oscillator length

$$b(\delta)=\sqrt{\frac{\hbar}{m\omega_0(\delta)}} \qquad \vec{r}\,'=\vec{r}/b$$

we get

$$H_0(\delta)=-\frac{\hbar^2}{2m}\frac{m\omega_0(\delta)}{\hbar}\Delta+\frac{m}{2}\left[\omega_0^{\,2}(\delta)\frac{\hbar}{m\omega_0}\left(1+\frac{2}{3}\delta\right)\left(x^2+y^2\right)+\omega_0^{\,2}(\delta)\frac{\hbar}{m\omega_0}\left(1-\frac{4}{3}\delta\right)z^2\right]$$

$$=-\frac{\hbar\omega_0(\delta)}{2}\Delta+\frac{\hbar\omega_0(\delta)}{2}\left[x^2+y^2+z^2+\frac{2}{3}\delta\left(x^2+y^2\right)-\frac{4}{3}\delta z^2\right]$$

$$=\frac{\hbar\omega_0(\delta)}{2}\left[-\Delta+r^2\right]-\delta\hbar\omega_0(\delta)\frac{4}{3}\sqrt{\frac{\pi}{5}}r^2Y_{20}(\Omega)$$

$$=\overset{0}{H}_0+H_\delta$$

Axial symmetry: cylindrical basis

$$\left\{N,n_z,n_\rho,m_\ell,m_s\right\}$$

$$N=n_x+n_y+n_z=n_z+2n_\rho+m_\ell$$

$$\varepsilon\left(n_z,n_\rho,m_\ell\right)=\sum_{i=x,y,z}\hbar\omega_i\left(n_i+\frac{1}{2}\right)=\hbar\omega_z\left(n_z+\frac{1}{2}\right)+\hbar\omega_\perp\left(2n_\rho+m_\ell+1\right)$$

$$=\hbar\overset{0}{\omega}_0\left[\left(N+\frac{3}{2}\right)+\delta\left(\frac{N}{3}-n_z\right)\right]$$

Anisotropic Harmonic Oscillator

Eigen-states characterized by $\Omega\pi\left[Nn_z m_\ell\right]$ $\quad \Omega = m_\ell \pm \frac{1}{2}$ $\quad \pi = (-1)^N$

$$\phi_{n_z n_\rho m_\ell m_s}\left(\vec{R},\sigma\right) = \psi_{n_\rho}^{m_\ell}(\rho)\,\psi_{n_z}(z)\,\frac{e^{im_\ell\varphi}}{\sqrt{2\pi}}\,\chi_{m_s}(\sigma)$$

$$\psi_{n_\rho}^{m_\ell}(\rho) \sim L_{n_\rho}^{m_\ell}(\rho)$$

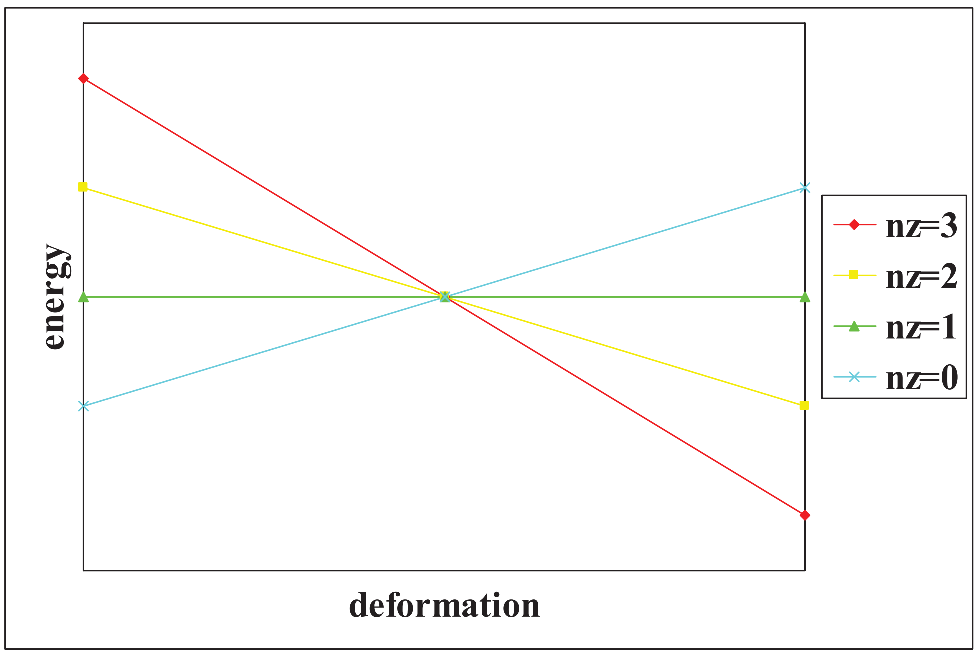
$$\psi_{n_z}(z) \sim H_{n_z}(z)$$

Anisotropic Harmonic Oscillator

Energy level structure: N=3

$$\varepsilon(n_z, n_\rho, m_\ell) = \sum_{i=x,y,z} \hbar\omega_i \left(n_z + \frac{1}{2}\right) = \hbar\omega_z\left(n_z + \frac{1}{2}\right) + \hbar\omega_\perp\left(2n_\rho + m_\ell + 1\right)$$
$$= \hbar\overset{0}{\omega}_0\left[\left(N + \frac{3}{2}\right) + \delta\left(\frac{N}{3} - n_z\right)\right]$$

$$\varepsilon^{N=3}(n_z n_\rho m_\ell) = \frac{9}{2}\hbar\overset{0}{\omega}_0 + \hbar\overset{0}{\omega}_0\,\delta(1 - n_z)$$



$$N = n_z + 2n_\rho + m_\ell$$

n_z	m_l	n_ρ	Ω	deg
0	3	0	5/2, 7/2	4
	1	1	1/2, 3/2	
1	2	0	3/2, 5/2	3
	0	1	1/2	
2	1	0	1/2, 3/2	2
3	0	0	1/2	1

The Nilsson model: Hamiltonian

Axially symmetric harmonic oscillator potential

+spin-orbit term

+l^2 term

$$H = H_0 + C\vec{\ell}\cdot\vec{s} + D\left(\vec{\ell}^2 - \left\langle \vec{\ell}^2 \right\rangle_N\right)$$

$$= \hbar\omega_0(\delta)\left[-\frac{1}{2}\Delta + \frac{1}{2}r^2 - \beta r^2 Y_{20}\right] - \kappa\hbar\overset{0}{\omega}_0\left[2\vec{\ell}\cdot\vec{s} + \mu\left(\vec{\ell}^2 - \left\langle \vec{\ell}^2 \right\rangle_N\right)\right]$$

$$C = -2\kappa\hbar\overset{0}{\omega}_0$$

$$D = -\kappa\mu\hbar\overset{0}{\omega}_0$$

$$\left\langle \vec{\ell}^2 \right\rangle_N = \frac{1}{2}N(N+3)$$

The Nilsson model: Hamiltonian

$$H = \overset{0}{H}_0 + \kappa\hbar\overset{0}{\omega}_0 F$$

$$F = \frac{\delta}{\kappa}\left[1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3\right]^{-1/6}\left\{-\frac{4}{3}\sqrt{\frac{\pi}{5}}r^2 Y_{20}\right\} - 2\vec{\ell}\cdot\vec{s} - \mu\left(\vec{\ell}^2 - \left\langle\vec{\ell}^2\right\rangle_N\right)$$

$$F = \eta U - 2\vec{\ell}\cdot\vec{s} - \mu\left(\vec{\ell}^2 - \left\langle\vec{\ell}^2\right\rangle_N\right)$$

$$E = \left(N + \frac{3}{2}\right)\hbar\omega_0(\delta) + \kappa\hbar\overset{0}{\omega}_0 f$$

	N,Z<50	50<Z<82	82<N<126	82<Z	126<N
κ	0.08	0.0637	0.0637	0.0577	0.0635
μ	0	0.60	0.42	0.65	0.325

The Nilsson model: Basis

$\vec{\ell}\cdot\vec{s}$ and $\vec{\ell}^{\,2}$ nondiagonal in basis $\{N, n_z, n_\rho, m_\ell, m_s\}$

For large deformations $\vec{\ell}\cdot\vec{s}$, $\vec{\ell}^{\,2}$ can be neglected:

Asymptotic quantum numbers $\{N, n_z, n_\rho, m_\ell, m_s\}$: $[N n_z m_\ell]\Omega\pi$

For small deformations δ-terms can be neglected:

Spherical basis $\{N, \ell, j, \Omega\}$

Nilsson used basis $\{N, \ell, m_\ell, m_s\}$

Diagonal terms

$$\overset{0}{H}_0 \left| N, \ell, m_\ell, m_s \right\rangle = \left(N + \frac{3}{2} \right) \hbar \overset{0}{\omega}_0 \left| N, \ell, m_\ell, m_s \right\rangle$$

$$\vec{\ell}^{\,2} \left| N, \ell, m_\ell, m_s \right\rangle = \ell\left(\ell + 1\right) \left| N, \ell, m_\ell, m_s \right\rangle$$

The Nilsson model: Matrix elements

Matrix elements

$$\left\langle \ell' m'_\ell m'_s \left| \vec{\ell}\cdot\vec{s} \right| \ell m_\ell m_s \right\rangle \quad \begin{cases} \ell = \ell' \\ m_\ell = m'_\ell, m'_\ell \pm 1 \\ m_s = m'_s \pm 1, m'_s \\ m_\ell + m_s = m'_\ell + m'_s \end{cases}$$

$$\left\langle \ell, m_\ell \pm 1, \mp \left| \vec{\ell}\cdot\vec{s} \right| \ell, m_\ell, \pm \right\rangle = \frac{1}{2}\sqrt{(\ell \mp m_\ell)(\ell \pm m_\ell + 1)}$$

$$\left\langle \ell, m_\ell, \pm \left| \vec{\ell}\cdot\vec{s} \right| \ell, m_\ell, \pm \right\rangle = \pm\frac{1}{2} m_\ell$$

$$\left\langle \ell' m'_\ell \left| Y_{20} \right| \ell m_\ell \right\rangle = i^{\ell-\ell'} \sqrt{\frac{5}{4\pi}} \sqrt{\frac{2\ell+1}{2\ell'+1}} \left\langle \ell 2 m_\ell 0 \middle| \ell 2 \ell' m'_\ell \right\rangle \left\langle \ell 2 0 0 \middle| \ell 2 \ell' 0 \right\rangle$$

$$\begin{cases} m_\ell = m'_\ell \quad m_s = m'_s \\ \ell = \ell', \ell' \pm 2 \quad N = N' \pm 2 \end{cases}$$

The Nilsson model: Matrix elements

Radial matrix elements

$$\left|N\ell\right\rangle = \sqrt{\frac{2(n-1)!}{b^3\left[\Gamma(n+\ell+1/2)\right]^3}}\left(\frac{r}{b}\right)^{\ell} e^{-\frac{1}{2}\left(\frac{r}{b}\right)^2} L_{n-1}^{\ell+1/2}\left(r^2/b^2\right)$$

$$\left\langle N'\ell'\middle|r^2\middle|N\ell\right\rangle = \left[\frac{(n'-1)!(n-1)!}{\left[\Gamma(n'+\ell'+1/2)\right]\left[\Gamma(n+\ell+1/2)\right]}\right]^{1/2} b^2(-1)^{n'+n}\mu!\nu!$$

$$\times\sum_{\sigma}\frac{\Gamma(p+\sigma+1)}{\sigma!(n'-1-\sigma)!(n-1-\sigma)!(\sigma+\mu-n'+1)!(\sigma+\nu-n+1)!}$$

$p=\frac{1}{2}(\ell+\ell'+3)$ $\mu=p-\ell'-1/2$ $\nu=p-\ell-1/2$

$N=2(n-1)+\ell$

$N, N\pm 2$ admixtures

$$\left\langle N\ell\middle|r^2\middle|N\ell\right\rangle=(2n+\ell-1/2)=(N+3/2)$$

$$\left\langle N\ell-2\middle|r^2\middle|N\ell\right\rangle=2\sqrt{n(n+\ell-1/2)}$$

$$\left\langle N-2\ell\middle|r^2\middle|N\ell\right\rangle=-\sqrt{(n-1)(n+\ell-1/2)}$$

$$\left\langle N-2\ell-2\middle|r^2\middle|N\ell\right\rangle=\sqrt{(n+\ell-1/2)(n+\ell-3/2)}$$

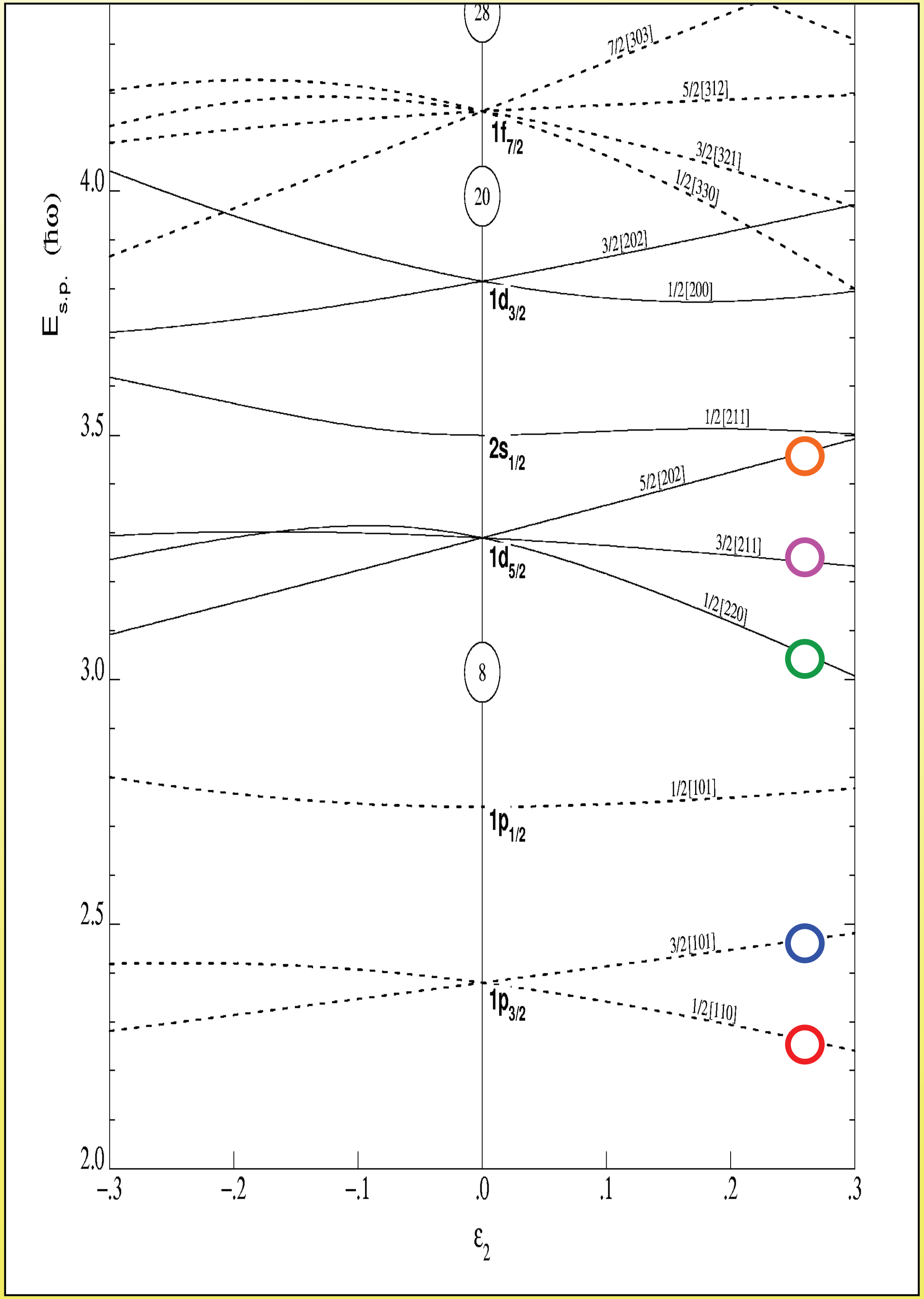
$$\left\langle N-2\ell+2\middle|r^2\middle|N\ell\right\rangle=\sqrt{(n-1)(n-2)}$$

The Nilsson model

Nilsson states:

$$\left|i\right\rangle_{\{\Omega\pi\}} = \sum_{\alpha} C_i^{\alpha} \left|\alpha\right\rangle; \quad \alpha\left\{N\ell m_\ell m_s\right\}$$

N	Ω	$\left\|N\ell m_\ell m_s\right\rangle_{\Omega\pi}$			
$N=0$	$\Omega=1/2$	$\left\|000+\right\rangle_{1/2^+}$			
$N=1$	$\Omega=3/2$	$\left\|111+\right\rangle_{3/2^-}$			
	$\Omega=1/2$	$\left\|110+\right\rangle_{1/2^-}$	$\left\|111-\right\rangle_{1/2^-}$		
$N=2$	$\Omega=5/2$	$\left\|222+\right\rangle_{5/2^+}$			
	$\Omega=3/2$	$\left\|221+\right\rangle_{3/2^+}$	$\left\|222-\right\rangle_{3/2^+}$		
	$\Omega=1/2$	$\left\|220+\right\rangle_{5/2^+}$	$\left\|200+\right\rangle_{5/2^+}$	$\left\|221-\right\rangle_{5/2^+}$	
$N=3$	$\Omega=7/2$	$\left\|333+\right\rangle_{7/2^-}$			
	$\Omega=5/2$	$\left\|332+\right\rangle_{5/2^-}$	$\left\|333-\right\rangle_{5/2^-}$		
	$\Omega=3/2$	$\left\|331+\right\rangle_{3/2^-}$	$\left\|311+\right\rangle_{3/2^-}$	$\left\|332-\right\rangle_{3/2^-}$	
	$\Omega=1/2$	$\left\|330+\right\rangle_{1/2^-}$	$\left\|331+\right\rangle_{3/2^-}$	$\left\|331-\right\rangle_{1/2^-}$	$\left\|311-\right\rangle_{1/2^-}$



$^{A}_{Z}Nucleus_{N} \rightarrow K^{\pi}$

$^{27}_{13}\mathrm{Al}_{14} \rightarrow 5/2^{+}$

$^{23}_{11}\mathrm{Na}_{12} \rightarrow 3/2^{+}$

$^{19}_{9}\mathrm{F}_{10} \rightarrow 1/2^{+}$

$^{9}_{4}\mathrm{Be}_{5} \rightarrow 3/2^{-}$

$^{7}_{3}\mathrm{Li}_{4} \rightarrow 1/2^{-}$

$^{27}_{14}\mathrm{Si}_{13} \rightarrow 5/2^{+}$

$^{25}_{12}\mathrm{Mg}_{13} \rightarrow 5/2^{+}$

$^{19}_{8}\mathrm{O}_{11} \rightarrow 3/2^{+}$

$^{19}_{10}\mathrm{Ne}_{9} \rightarrow 1/2^{+}$

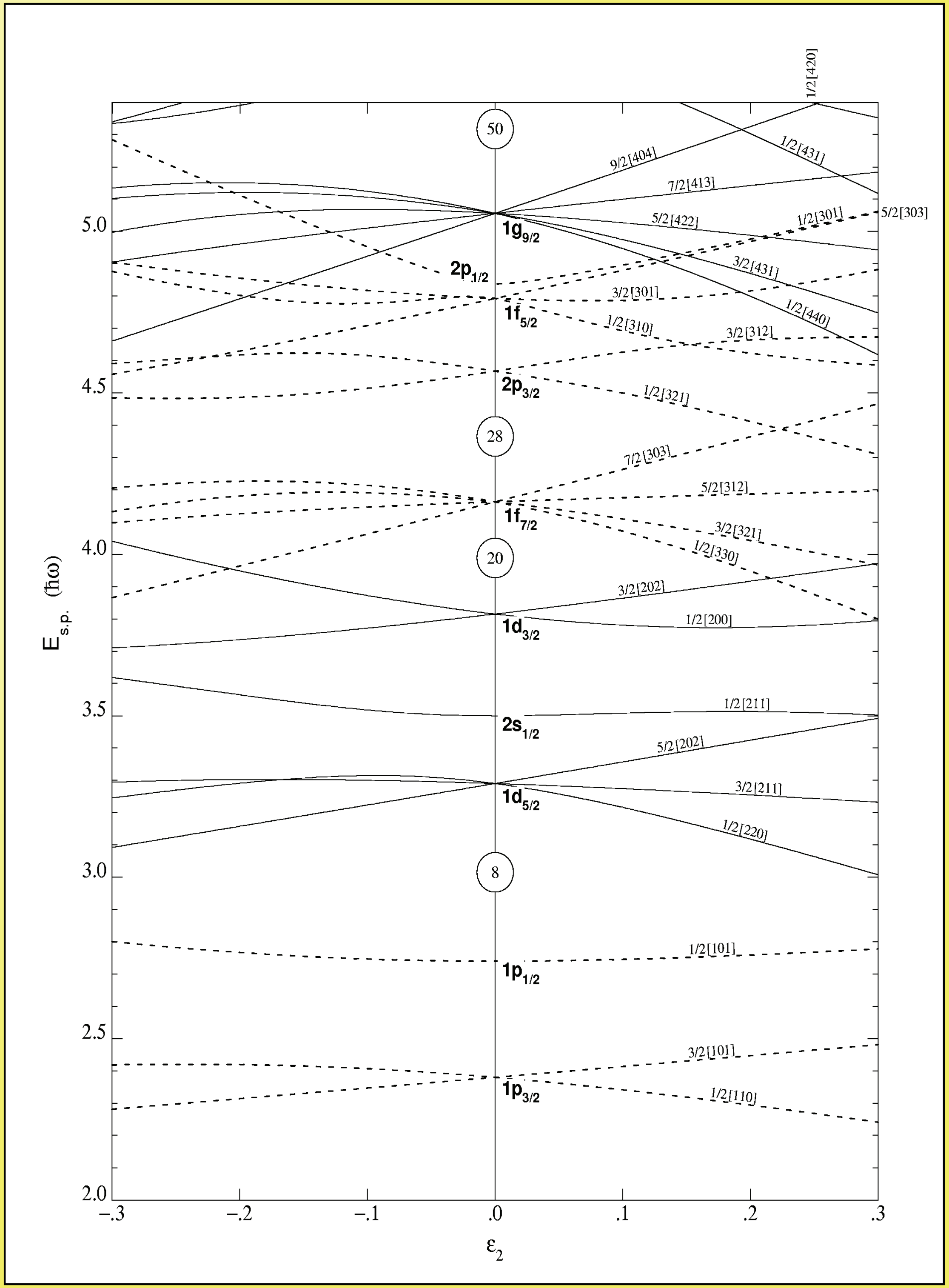
- Spherical levels split into (2j+1)/2 levels

- Levels ($\Omega\pi$) are twofold degenerate

- Asymptotic q-numbers not conserved for small deformations but useful to classify levels

- For positive deformations (PROLATE SHAPES), levels with lower Ω are shifted downwards

- For negative deformations (OBLATE SHAPES), levels with lower Ω are shifted upwards



$$N = n_z + 2n_\rho + m_\ell$$

	$[Nn_z m_\ell]\Omega^\pi$

$N=0$	$\Omega=1/2^+$	$n_z=0$	$m_\ell=0$	$[000]1/2^+$
$N=1$	$\Omega=1/2^-$	$n_z=1$	$m_\ell=0$	$[110]1/2^-$
		$n_z=0$	$m_\ell=1$	$[101]1/2^-$
$N=1$	$\Omega=3/2^-$	$n_z=0$	$m_\ell=1$	$[101]3/2^-$
$N=2$	$\Omega=1/2^+$	$n_z=2$	$m_\ell=0$	$[220]1/2^+$
		$n_z=1$	$m_\ell=1$	$[211]1/2^+$
		$n_z=0$	$m_\ell=0$	$[200]1/2^+$
$N=2$	$\Omega=3/2^+$	$n_z=1$	$m_\ell=1$	$[211]3/2^+$
		$n_z=0$	$m_\ell=2$	$[202]3/2^+$
$N=2$	$\Omega=5/2^+$	$n_z=0$	$m_\ell=2$	$[202]5/2^+$
$N=3$	$\Omega=1/2^-$	$n_z=3$	$m_\ell=0$	$[330]1/2^-$
		$n_z=2$	$m_\ell=1$	$[321]1/2^-$
		$n_z=1$	$m_\ell=0$	$[310]1/2^-$
		$n_z=0$	$m_\ell=1$	$[301]1/2^-$
$N=3$	$\Omega=3/2^-$	$n_z=2$	$m_\ell=1$	$[321]3/2^-$
		$n_z=1$	$m_\ell=2$	$[312]3/2^-$
		$n_z=0$	$m_\ell=1$	$[301]3/2^-$
$N=3$	$\Omega=5/2^-$	$n_z=1$	$m_\ell=2$	$[312]5/2^-$
		$n_z=0$	$m_\ell=3$	$[303]5/2^-$
$N=3$	$\Omega=7/2^-$	$n_z=0$	$m_\ell=3$	$[303]7/2^-$