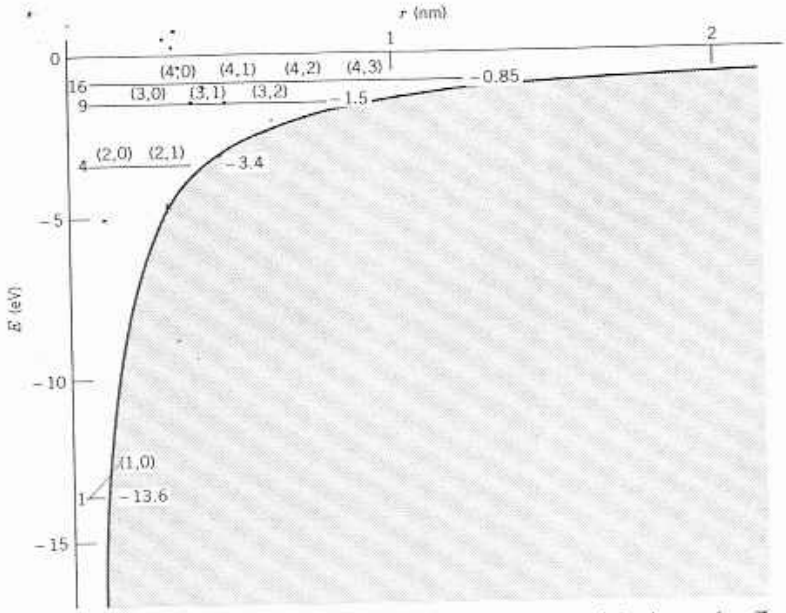

Figure 2.16 The lower energy levels in a Coulomb potential, shown for $Z = 1$ (hydrogen atom). The states are labeled with (n, ℓ); the degeneracies are indicated on the left and the energy values on the right.

nuclear models in Chapter 5. The behavior of angular momentum in quantum theory is discussed in the next section.

2.5 QUANTUM THEORY OF ANGULAR MOMENTUM

In solutions of the Schrödinger equation for three-dimensional problems, the quantum number ℓ plays a prominent role. In atomic physics, for example, it serves to label different electron wave functions and to tell us something about the spatial behavior of the wave functions. This *angular momentum quantum number* has the same function in all three-dimensional problems involving central potentials, where $V = V(r)$.

In classical physics, the angular momentum $\boldsymbol{\ell}$ of a particle moving with linear momentum $\boldsymbol{p}$ at a location $\boldsymbol{r}$ from a reference point is defined as

$$\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p} \tag{2.66}$$

In quantum mechanics, we can evaluate the expectation value of the angular momentum by analogy with Equation 2.10. We first consider the magnitude of the angular momentum, and for this purpose it is simplest to calculate ℓ^2. We must first find a quantum mechanical *operator* for ℓ^2, as we discussed in Section 2.2. This can be done simply by replacing the components of $\boldsymbol{p}$ with their operator equivalents: $p_x = -ih\,\partial/\partial x$, $p_y = -ih\,\partial/\partial y$, $p_z = -ih\,\partial/\partial z$. Evaluating the cross product then gives terms of the form $\ell_x = yp_z - zp_y$, and finally computing $\langle \ell^2 \rangle = \langle \ell_x^2 + \ell_y^2 + \ell_z^2 \rangle$ gives the remarkably simple result, which is

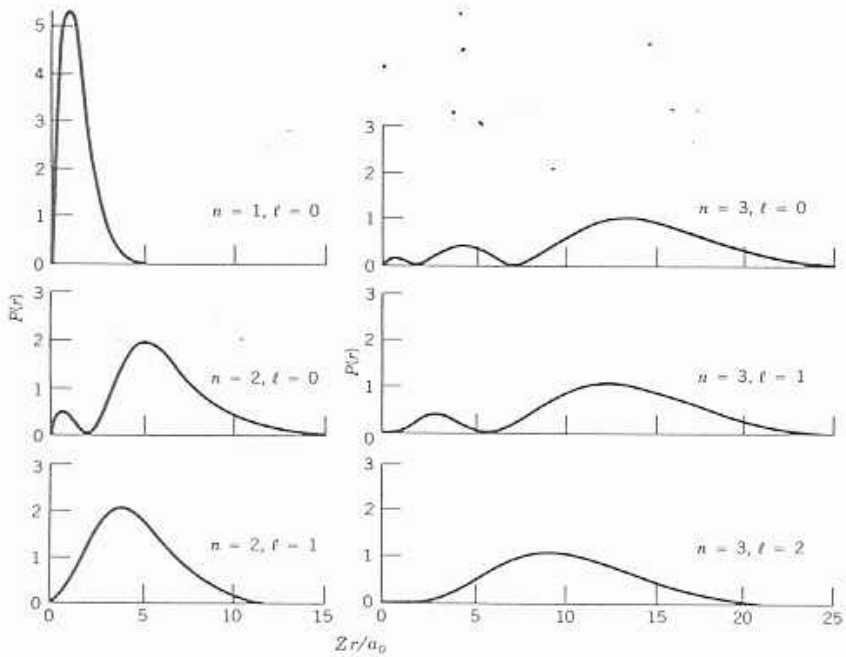

Figure 2.17 Radial probability distributions for a particle in a Coulomb potential (hydrogenic atom). The probability vanishes at $r = 0$, but as before the $\ell = 0$ wave functions do not. This property becomes especially important for phenomena that depend on the overlap of atomic wave functions with the nucleus — only $\ell = 0$ states contribute substantially to such phenomena (electron capture, hyperfine structure, etc.). Why doesn't the "centrifugal repulsion" appear to occur in this case?

independent of the form of $R(r)$,

$$\langle \ell^2 \rangle = \hbar^2 \ell(\ell + 1) \tag{2.67}$$

That is, whenever we have a central potential, which gives a wave function $R(r)Y_{\ell m_\ell}(\theta, \phi)$, the magnitude of the angular momentum is fixed at the value given by Equation 2.67; *the angular momentum is a constant of the motion* (as it is in classical physics for central potentials). The atomic substates with a given ℓ value are labeled using *spectroscopic notation*; we use the same spectroscopic notation in nuclear physics: s for $\ell = 0$, p for $\ell = 1$, and so on. These are summarized in Table 2.6.

When we now try to find the direction of ℓ, we run into a barrier imposed by the uncertainty principle: quantum mechanics permits us to know exactly only

Table 2.6 Spectroscopic Notation

ℓ value	0	1	2	3	4	5	6
Symbol	s	p	d	f	g	h	i

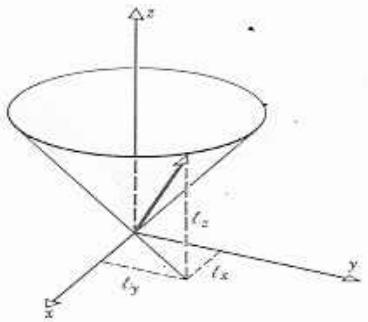

Figure 2.18 The vector ℓ precesses rapidly about the z axis, so that ℓ_z stays constant, but ℓ_x and ℓ_y are variable.

one component of ℓ at a time. Once we determine the value of one component, the other two components are completely indeterminate. (This is a fundamental limitation, and no amount of trickery can get us around it. It is the very act of measuring one component that *makes* the other two indeterminate. When we measure ℓ_x, we force ℓ_y and ℓ_z into indeterminacy; when we then measure ℓ_y for the same system, our previous knowledge of ℓ_x is destroyed as ℓ_x is now forced into indeterminacy.) By convention, we usually choose the z component of ℓ to be determined, and computing $\langle \ell_z \rangle$ as described above,

$$\langle \ell_z \rangle = \hbar m_\ell \tag{2.68}$$

where $m_\ell = 0, \pm 1, \pm 2, \ldots, \pm \ell$. Notice that $|\langle \ell_z \rangle| < |\ell| = \hbar\sqrt{\ell(\ell+1)}$—the z component of the vector is always less than its length. If $|\langle \ell_z \rangle| = |\ell|$ were permitted, then we would have exact knowledge of all three components of ℓ (ℓ_x and ℓ_y would be zero if ℓ were permitted to align with the z axis). The conventional vector representation of this indeterminacy is shown in Figure 2.18 —ℓ rotates or precesses about the z axis keeping ℓ_z fixed but varying ℓ_x and ℓ_y.

The complete description of an electronic state in an atom requires the introduction of a new quantum number, the *intrinsic angular momentum* or *spin*. For the electron, the spin quantum number is $s = \frac{1}{2}$. The spin can be treated as an angular momentum (although it cannot be represented in terms of classical variables, because it has no classical analog). Thus

$$\langle s^2 \rangle = \hbar^2 s(s+1) \tag{2.69}$$

$$\langle s_z \rangle = \hbar m_s \qquad \left(m_s = \pm \tfrac{1}{2}\right) \tag{2.70}$$

It is often useful to imagine the spin as a vector s with possible z components $\pm \frac{1}{2}\hbar$.

Nucleons, like electrons, have spin quantum numbers of $\frac{1}{2}$. A nucleon moving in a central potential with orbital angular momentum ℓ and spin s has a *total angular momentum*

$$j = \ell + s \tag{2.71}$$

The total angular momentum j behaves in a manner similar to ℓ and s:

$$\langle j^2 \rangle = \hbar^2 j(j+1) \tag{2.72}$$

$$\langle j_z \rangle = \langle \ell_z + s_z \rangle = \hbar m_j \tag{2.73}$$

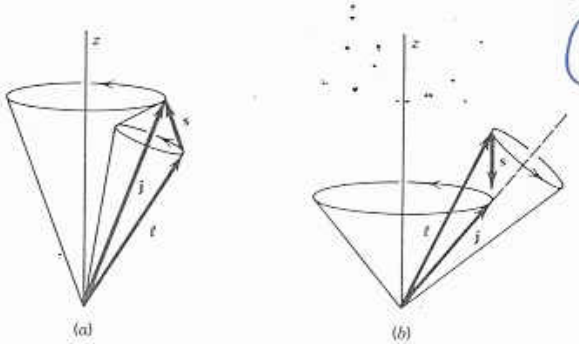

Figure 2.19 The coupling of orbital angular momentum ℓ to spin angular momentum s giving total angular momentum j. (*a*) Coupling giving $j = \ell + \frac{1}{2}$. The vectors ℓ and s have definite lengths, as does j. The combined ℓ and s vectors rotate or precess about the direction of j; in this coupling the z components ℓ_z and s_z thus do not have definite values. The vector j precesses about the z direction so that j_z has a definite value. (*b*) The similar case of $j = \ell - \frac{1}{2}$. In interpreting both figures, keep in mind that all such representations of vectors governed by the rules of quantum mechanics are at best symbolic and at worst misleading.

where $m_j = -j, -j+1, \ldots, j-1, j$ and where j is the total angular momentum quantum number. From Equations 2.68, 2.70, and 2.73 it is apparent that

$$m_j = m_\ell + m_s = m_\ell \pm \tfrac{1}{2} \tag{2.74}$$

Since m_ℓ is always an integer, m_j must be half-integral ($\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$) and thus j must be half-integral. The vector coupling of Equation 2.71 suggests only two possible values for j: $\ell + \frac{1}{2}$ or $\ell - \frac{1}{2}$, which are illustrated in Figure 2.19.

Usually, we indicate the j value as a subscript in spectroscopic notation. Thus, for $\ell = 1$ (p states), there are two possible j values: $\ell + \frac{1}{2} = \frac{3}{2}$ and $\ell - \frac{1}{2} = \frac{1}{2}$. We would indicate these states as $p_{3/2}$ and $p_{1/2}$. When there is an additional quantum number, such as a principal quantum number n (or perhaps just an index which counts the states in order of increasing energy), we indicate it as $2p_{3/2}$, $3p_{3/2}$, and so on.

In atoms, it is often useful for us to picture the electrons as moving in well defined orbits with definite ℓ and j. It is not at all obvious that a similar picture is useful for nucleons inside the nucleus, and thus it is not clear that ℓ and j will be useful labels. We discuss this topic in detail when we consider the nuclear shell model in Chapter 5.

2.6 PARITY

The parity operation causes a reflection of all of the coordinates through the origin: $r \to -r$. In Cartesian coordinates, this means $x \to -x$, $y \to -y$, $z \to -z$; in spherical coordinates, $r \to r$, $\theta \to \pi - \theta$, $\phi \to \phi + \pi$. If a system is left unchanged by the parity operation, then we expect that none of the observable properties should change as a result of the reflection. Since the values we measure for the observable quantities depend on $|\psi|^2$, then we have the following reasonable assertion:

$$\text{If } V(r) = V(-r), \quad \text{then } |\psi(r)|^2 = |\psi(-r)|^2.$$

This assertion, whose reverse is also true, has two important consequences for our work in nuclear physics:

1. If $|\psi(\boldsymbol{r})|^2 = |\psi(-\boldsymbol{r})|^2$ then $\psi(-\boldsymbol{r}) = \pm\psi(\boldsymbol{r})$. That is, the parity operation has either of two effects on a wave function. The case $\psi(-\boldsymbol{r}) = +\psi(\boldsymbol{r})$ is known as *positive* or *even* parity, while the case $\psi(-\boldsymbol{r}) = -\psi(\boldsymbol{r})$ is *negative* or *odd* parity. If the potential $V(\boldsymbol{r})$ is left unchanged by the parity operation, then the resulting stationary-state wave functions must be of either even or odd parity. Mixed-parity wave functions are not permitted. Recall our solutions for the one-dimensional harmonic oscillator. The potential $\frac{1}{2}kx^2$ is certainly invariant with respect to the parity operation $x \to -x$. The wave functions listed in Table 2.1 have either only odd powers of x, and therefore odd parity, or only even powers of x, and therefore even parity. Polynomials mixing odd and even powers do not occur. Also, review the solutions for the finite potential well. Since the well lies between $x = +a/2$ and $x = -a/2$, the potential is symmetric with respect to the parity operation: $V(x) = V(-x)$. Notice the solutions illustrated in Figure 2.8. For some of the solutions, $\psi(-x) = \psi(x)$ and their parity is even; the other solutions have $\psi(-x) = -\psi(x)$ and odd parity.

   In three dimensions, the parity operation applied to the $Y_{\ell m_\ell}$ gives a phase $(-1)^\ell$:

$$Y_{\ell m_\ell}(\pi - \theta, \phi + \pi) = (-1)^\ell Y_{\ell m_\ell}(\theta, \phi) \tag{2.75}$$

   Central potentials, which depend only on the magnitude of $\boldsymbol{r}$, are thus invariant with respect to parity, and their wave functions have definite parity, odd if ℓ is odd and even if ℓ is even.

   The wave function for a system of many particles is formed from the product of the wave functions for the individual particles. The parity of the combined wave function will be even if the combined wave function represents any number of even-parity particles or an even number of odd-parity particles; it will be odd if there is an odd number of odd-parity particles. Thus nuclear states can be assigned a definite parity, odd or even. This is usually indicated along with the total angular momentum for that state, as for example, $\frac{5}{2}^+$ or $\frac{3}{2}^-$. In Chapter 10 we will discuss how the parity of a state can be determined experimentally.

2. The second consequence of the parity rule is based on its converse. If we find a system for which $|\psi(\boldsymbol{r})|^2 \neq |\psi(-\boldsymbol{r})|^2$, then we must conclude that $V(\boldsymbol{r}) \neq V(-\boldsymbol{r})$; that is, the system is *not* invariant with respect to parity. In 1957 it was discovered that certain nuclear processes (β decays) gave observable quantities whose measured values did not respect the parity symmety. On the other hand, no evidence has yet been obtained that either the strong nuclear interaction or the electromagnetic interaction violate parity. The establishment of parity violation in β decay was one of the most dramatic discoveries in nuclear physics and has had profound influences on the development of theories of fundamental interactions between particles. A description of these experiments is given in Section 9.9.

2.7 QUANTUM STATISTICS

When we group several particles together to make a larger quantum system (several nucleons in a nucleus, several electrons in an atom, several atoms in a molecule) a new quantum effect arises if the particles are indistinguishable from one another. Let us consider the case of two particles, for example, the two electrons in a helium atom. Suppose one electron is described by coordinates r_1 and is in the state ψ_A, while the other electron is described by coordinates r_2 and is in the state ψ_B. The combined wave function is the product of the two component wave functions; thus $\psi = \psi_A(r_1)\psi_B(r_2)$. Now suppose the two electrons are interchanged so that the new wave function is $\psi' = \psi_B(r_1)\psi_A(r_2)$. Is there any measurement we could do to detect whether this interchange had taken place?

If the electrons are truly *indistinguishable*, the answer to this question must be no. There is no *observational scheme* for distinguishing the "first electron" from the "second electron." Thus we have a result that is somewhat similar to our result for the parity operation: *Probability densities must be invariant with respect to exchange of identical particles*. That is, the exchanged wave function ψ_{21} can at most differ only in sign from the original wave function ψ_{12}. We therefore have two cases. If the sign does not change upon exchange of the particles, we have a *symmetric* wave function; for symmetric wave functions, $\psi_{12} = \psi_{21}$. If the exchange changes the sign, we have an *antisymmetric* wave function, for which $\psi_{21} = -\psi_{12}$. *All combined wave functions representing identical particles must be either completely symmetric or completely antisymmetric*. No "mixed symmetry" wave functions are allowed.

When we turn to our laboratory experiments to verify these assertions, we find a further classification to which there are no known exceptions: all particles with integral spins $(0, 1, 2, \ldots)$ have symmetric combined wave functions, while all particles with half-integral spins $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots)$ have antisymmetric combined wave functions.

The above two-particle functions ψ and ψ' will not do for combined wave functions because they are neither symmetric nor antisymmetric. That is, ψ' does not at all look like either ψ or $-\psi$. Instead, consider the following combined wave function:

$$\psi_{12} = \frac{1}{\sqrt{2}}\left[\psi_A(r_1)\psi_B(r_2) \pm \psi_B(r_1)\psi_A(r_2)\right] \tag{2.76}$$

If we choose the plus sign, then the combined wave function is symmetric with respect to interchange of the particles. If we choose the minus sign then the result is an antisymmetric wave function. The factor of $1/\sqrt{2}$ ensures that the resulting combination is normalized (assuming that each of the component wave functions is itself normalized).

A special case arises when we have identical quantum states A and B. (We can regard A and B as representing a set of quantum numbers.) When A is the same as B, the antisymmetric combination wave function vanishes identically, and so its probability density is always zero. *The probability to find two identical particles of half-integral spin in the same quantum state must always vanish*. This is of course

(7)

just the *Pauli exclusion principle*, which determines why atomic subshells fill in a certain way. This vanishing of the antisymmetric wave function is the mathematical basis of the Pauli principle. No such vanishing occurs for the symmetric combination, so there is nothing to prevent identical particles of integral spin from occupying the same quantum state.

Later in this text, we apply the Pauli principle to nucleons and show its importance in understanding the nuclear shell model. We also construct some simple antisymmetric wave functions for the quarks that make up nucleons and other similar particles.

2.8 TRANSITIONS BETWEEN STATES

A true stationary state lives forever. The expectation values of physical observables, computed from the wave function of a stationary state, do not change with time. In particular, the expectation value of the energy is constant in time. The energy of the state is precisely determined, and the uncertainty in the energy,

$$\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2} \tag{2.77}$$

vanishes, because $\langle E^2 \rangle = \langle E \rangle^2$ for this case. The Heisenberg relationship, $\Delta E \Delta t \geq \hbar/2$, then implies that $\Delta t = \infty$. Thus a state with an exact energy lives forever; its lifetime against decay (to lower excited states, for example) is infinite.

Now suppose our system is subject to a weak perturbing potential V', in addition to the original potential V. In the absence of V', we can solve the Schrödinger equation for the potential V and find a set of eigenstates ψ_n and corresponding eigenvalues E_n. If we now include the weak additional potential V', we find that the states are approximately, but not exactly, the previous eigenstates ψ_n of V. This weak additional potential permits the system to make transitions between the "approximate" eigenstates ψ_n. Thus, under the interaction with a weak electromagnetic field, a hydrogen atom can make transitions, such as $2p \rightarrow 1s$ or $3d \rightarrow 2p$. We still describe the various levels as if they were eigenstates of the system.

Even though a system may make a transition from an initial energy state E_i to a final state E_f, energy must be conserved. Thus the total decay energy must be constant. If the final state E_f is of lower energy than E_i, the energy difference $E_i - E_f$ must appear as radiation emitted in the decay. In transitions between atomic or nuclear excited states, a photon is emitted to carry the energy $E_i - E_f$.

A nonstationary state has a nonzero energy uncertainty ΔE. This quantity is often called the "width" of the state and is usually represented by Γ. The *lifetime* τ of this state (the mean or average time it lives before making a transition to a lower state) can be estimated from the uncertainty principle by associating τ with the time Δt during which we are permitted to carry out a measurement of the energy of the state. Thus $\tau = \hbar/\Gamma$. The *decay probability* or *transition probability* λ (the number of decays per unit time) is inversely related to the mean lifetime τ:

$$\lambda = \frac{1}{\tau} \tag{2.78}$$

It would be very useful to have a way to calculate λ or τ directly from the nuclear wave functions. We can do this if we have knowledge of (1) the initial

and final wave functions ψ_i and ψ_f, which we regard as approximate stationary states of the potential V; and (2) the interaction V' that causes the transition between the states. The calculation of λ is too detailed for this text, but can be found in any advanced text on quantum mechanics. We will merely state the result, which is known as *Fermi's Golden Rule*:

$$\lambda = \frac{2\pi}{h}|V'_{fi}|^2\rho(E_f) \tag{2.79}$$

The quantity V'_{fi} has the form of an expectation value:

$$V'_{fi} = \int \psi_f^* V' \psi_i \, dv \tag{2.80}$$

Notice again the ordering of the states f and i in the integral. The integral V'_{fi} is sometimes called the *matrix element* of the transition operator V'. This terminology comes from an alternative formulation of quantum mechanics based on matrices instead of differential equations. Be sure to take special notice that *the decay probability depends on the square of the transition matrix element*.

The quantity $\rho(E_f)$ is known as the *density of final states*. It is the number of states per unit energy interval at E_f, and it must be included for the following reason: if the final state E_f is a single isolated state, then the decay probability will be much smaller than it would be in the case that there are many, many states in a narrow band near E_f. If there is a large density of states near E_f, there are more possible final states that can be reached by the transition and thus a larger transition probability. The density of final states must be computed based on the type of decay that occurs, and we shall consider examples when we discuss β decay, γ decay, and scattering cross sections.

REFERENCES FOR ADDITIONAL READING

The following introductory (sophomore-junior) modern physics texts provide background material necessary for the study of quantum mechanics: A. Beiser, *Concepts of Modern Physics*, 3rd ed. (New York: McGraw-Hill, 1981); K. S. Krane, *Modern Physics* (New York: Wiley, 1983); P. A. Tipler, *Modern Physics* (New York: Worth, 1978); R. T. Weidner and R. L. Sells, *Elementary Modern Physics*, 3rd ed. (Boston: Allyn and Bacon, 1980).

Quantum mechanics references at about the same level as the present text are listed below: R. Eisberg and R. Resnick, *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles*, 2nd ed. (New York: Wiley, 1985); A. P. French and E. F. Taylor, *An Introduction to Quantum Physics* (New York: Norton, 1978); R. B. Leighton, *Principles of Modern Physics* (New York: McGraw-Hill, 1969); D. S. Saxon, *Elementary Quantum Mechanics* (San Francisco: Holden-Day, 1968).

Advanced quantum texts, which can be consulted to find more detailed discussions of topics discussed only briefly in this text, are the following: C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics* (New York: Wiley-Interscience, 1977); D. Park, *Introduction to the Quantum Theory*, 2nd ed. (New York: McGraw-Hill, 1974); E. Merzbacher, *Quantum Mechanics*, 2nd ed. (New York: Wiley, 1970).

Repaso: espín y matrices de Pauli

$$\sigma_1 \,(\acute{o}\ \sigma_x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2\,(\sigma_y) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \sigma_z$$

Cumplen las relaciones del álgebra $SU(2)$

$$\left.\begin{array}{ll} \sigma_1^2 = \mathbb{I}_{2\times 2} & [\sigma_1, \sigma_2] = 2i\sigma_3 \\ \sigma_2^2 = \mathbb{I}_{2\times 2} & [\sigma_2, \sigma_3] = 2i\sigma_1 \\ \sigma_3^2 = \mathbb{I}_{2\times 2} & [\sigma_3, \sigma_3] = -2i\sigma_2 \end{array}\right\} [\sigma_i, \sigma_j] = 2i\epsilon_{jke}\sigma_e$$

Son una representación del operador momento angular para dimensión 2, esto es, $j = 1/2$, en este caso se usa para el espín $\frac{1}{2}$ (e isoespín $\frac{1}{2}$).

Espinores: $\chi^{\uparrow} \equiv |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mathbb{I}_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad \chi^{\downarrow} \equiv |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

[representación matricial]

$$\vec{S} = \frac{1}{2}\hbar\,\vec{\sigma}$$

Por ejemplo: $\vec{S}^2 = \frac{1}{4}\hbar^2\,\vec{\sigma}\cdot\vec{\sigma} = \frac{1}{4}\hbar^2\, 3\,\mathbb{I}_{2\times 2}$ ($\vec{\sigma}\cdot\vec{\sigma} = \sigma_x^2+\sigma_y^2+\sigma_z^2$)

por tanto: $\vec{S}^2\chi^{\uparrow} = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4}\hbar^2$

y lo mismo para $\chi^{\downarrow}$

También: $S_z = \frac{1}{2}\hbar\,\sigma_z$ $\qquad S_z\chi^{\uparrow} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\hbar\begin{pmatrix} 1 \\ 0 \end{pmatrix} =$

$$= +\frac{1}{2}\hbar$$

$$S_z\chi^{\downarrow} = \frac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\hbar$$

Más repaso: estado singlete y triplete para dos partículas de espín $\frac{1}{2}$.

$S=0$, SINGLETE: $\frac{1}{\sqrt{2}}\left\{\chi_1^{\uparrow}\chi_2^{\downarrow} - \chi_1^{\downarrow}\chi_2^{\uparrow}\right\} =$

es antisimétrico ante el intercambio de los espines $= \frac{1}{\sqrt{2}}\left\{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right\}$

$S=1$, TRIPLETE

$S_z = +1$, $\quad \chi_1^{\uparrow}\chi_2^{\uparrow} = |\uparrow\uparrow\rangle$

$S_z = 0$ $\quad \frac{1}{\sqrt{2}}\left\{\chi_1^{\uparrow}\chi_2^{\downarrow} + \chi_1^{\downarrow}\chi_2^{\uparrow}\right\} =$

$= \frac{1}{\sqrt{2}}\left\{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right\}$

$S_z = -1$ $\quad \chi_1^{\downarrow}\chi_2^{\downarrow} = |\downarrow\downarrow\rangle$

Lo mismo para el isoespín ($T=0$ ó $T=1$)

. Ejercicio (para entregar el lunes 28 de Octubre a las 9:30 en el aula 10 o en mi casillero)

Probar que: $\langle S | \vec{S}_1 - \vec{S}_2 | S \rangle = 0$, para $S=0$ y para $S=1$

Fig. 5 shows observed nuclides. For small numbers of nucleons, the band is nearly diagonal, *i.e.*, $Z \approx N$. As the size grows, the band bends and is below the diagonal, $Z < N$. Using the empirical mass formula, the existence of the band is easy to understand. As you go away from the band, the symmetry term becomes important and the mass of the nucleus grows. What it means it that such a nuclide, if exists, decays immediately by ejecting excess neutrons or protons until the symmetry term becomes small enough to make it energetically impossible to eject free neutrons or protons. For large nuclei, Coulomb term is important and smaller number of protons is preferred. That is why the band bends downwards. Even within the band, the number of stable nuclides is not so large. All the colored ones decay either by β-decay $(N, Z) \to (N-1, Z+1)e^-\bar{\nu}_e$ or anti-β-decay $(N, Z) \to (N+1, Z-1)e^+\nu_e$ to approach the narrow band of stability moving along $-45°$ line. Unstable nuclei can also emit an α-particle, a unusually tightly bound ^{4}He nucleus, to lower the mass number, approaching the maximum binding energy of $A = 56$.

3 Nuclear Force

Protons and neutrons are bound inside nuclei, despite the Coulomb repulsion among protons. Therefore there must be a different and much stronger force acting among nucleons to bind them together. This force is called nuclear force, nuclear binding force, or in more modern settings, *the strong interaction*. (Here, we are not talking about *a* strong interaction. This is the name of the force.) Here are notable properties of the nuclear binding force.

1. It is much stronger than the electromagnetic force. In the empirical mass formula, we saw that the coefficient of the Coulomb term is more than an order of magnitude smaller than the other terms in the binding energy.
2. It is an attractive force, otherwise nucleons wouldn't bind.
3. It is short-ranged, acts only up to 1–2 fm.
4. It has the saturation property, giving nearly constant $B/A \simeq 8.5$ MeV. This is in stark contrast to the electromagnetic force. For instance, the Thomas–Fermi model of atoms gives $B = 15.73Z^{7/3}$ eV that grows with a very high power in the number of particles.

(12)

5. The force depends on spin and charge states of the nucleon. To understand nuclei and nucleon-nucleon scattering data, we need not only a potential $V(r)$ between nucleons in the Hamiltonian but also the spin-spin term $\vec{\sigma}_1 \cdot \vec{\sigma}_2 V(r)$, the spin-orbit term $(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{L} V(r)$, and the tensor term $[3(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r}) - r^2 \vec{\sigma}_1 \cdot \vec{\sigma}_2] V(r)$.

6. It can exchange charge. If you do neutron-proton scattering experiment, you not only see a forward peak but also a backward peak. Note that a forward peak is analogous to a large impact parameter in the classical mechanics where there is little deflection (recall Rutherford scattering), and exists for pretty much any scattering processes. But a backward peak is quite unusual. The interpretation is that when the proton appears to be backscattered, it is actually a neutron which converted to a proton because of the nuclear reaction. In other words, the neutron is scattered to the forward angle, but has converted to proton by the scattering and we are fooled to see the proton scattered backward. This is the charge-exchange reaction.

7. Even though the nuclear force is attractive to bind nucleons, there is a repulsive core when they approach too closely, around 0.5 fm. They basically cannot go closer.

8. The nuclear force has "charge symmetry," which means that we can make an overall switch between protons and neutrons without changing forces among them. For instance, nn and pp scattering are the same (except for the obvious difference due to the electric charge). For example, "mirror nuclei," which are related by switching protons and neutrons, have very similar excitation spectra. Examples include ^{13}C and ^{13}N, ^{17}O and ^{17}F, etc.

9. A stronger version of the charge symmetry is "charge independence." Not only nn and pp scattering are the same, but also np scattering is also the same under the "same configuration" which I specify below using the concept of isospin.

The last item needs some more explanations. There is a new symmetry in the nuclear force called isospin, proposed originally by Heisenberg. The idea is very simple: regard protons and neutrons as identical particles. But of course, you can't; they are different particles, right? They even have different

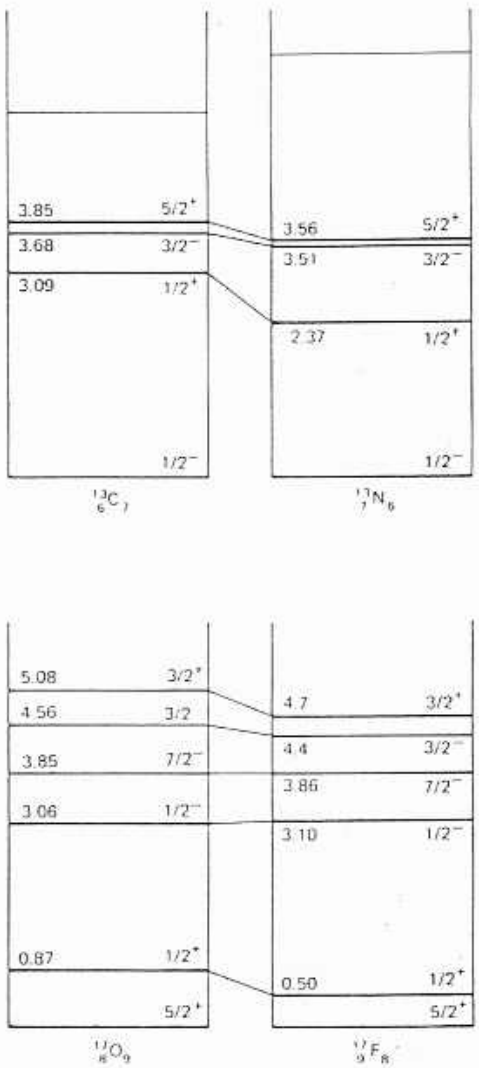

Figure 7: Comparison of excitation spectrum of two mirror nuclei, ^{13}C and ^{13}N, ^{17}O and ^{17}F. From "Theoretical Nuclear Physics," by Amos deShalit and Herman Feshbach, New York, Wiley, 1974.

masses! Well, the trick is to introduce a new quantum number, isospin, which takes values $+1/2$ and $-1/2$ just like the ordinary spin. We say a proton is a nucleon with $I_z = +1/2$, while a neutron with $I_z = -1/2$. At this point, it is just semantics. But the important statement is this: the nuclear force is invariant under the isospin rotation, just like the Hamiltonian of a ferromagnet is invariant under the rotation of spin. Then you can classify states according to the isospin quantum numbers because the nuclear force preserves isospin. But what about the mass difference, then? The point is that their masses are actually quite similar: $m_p = 938.3$ MeV/c^2 and $m_n = 939.6$ MeV/c^2. To the extent that we ignore the small mass difference, we can treat them identical. Another question is the obvious difference in their electric charges $+|e|$ and 0. Again, the Coulomb force is not the dominant force in nuclei, as we have seen in the empirical mass formula. We can ignore the difference in the electric charge and put it back in as a "small" perturbation.

The charge symmetry is a limited example of the isospin invariance. It corresponds to the overall reversal of all isospins. If you reverse all spins s_z, that is basically the 180° rotation around the y-axis, and you obtain another state with degenerate energy. Likewise, if you reverse all isospins, by rotating the isospin around the "isospin y-axis" by 180°, you interchange protons with neutrons, just like interchanging spin up and spin down states. If the nuclear force is indeed invariant under the isospin rotation, it must also be invariant under the isospin reversal. Fig. 7) shows that indeed the nuclear spectra approximately respect this invariance. Of course, isospin is not an exact symmetry because protons and neutrons have different electric charges. But the isospin invariance goes even further ("charge independence"). It says that not only the interaction between pp and nn are the same ("charge symmetry"), also np is, except that you have to carefully select the configuration. Here is what is required. Because proton and neutron both carry $T = 1/2$ (and opposite $T_z = \pm 1/2$), two nucleon states would have both $T = 1$ and $T = 0$ components. Both pp and nn states are [illegible] in the $T = 1$ state. On the other hand, the np state can either be in the $T = 1$ or $T = 0$ states. But the fermion wave function must be anti-symmetric while $T = 1$ ($T = 0$) isospin wave function is symmetric (anti-symmetric). Therefore, if the space and spin wave function of a np state is symmetric (anti-symmetric), it selects $T = 0$ ($T = 1$) isospin wave function. This way, you can separate purely $T = 1$ part of the np wave function, and compare the interaction to that of the nn and pp states. And they are indeed the same up to corrections

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(en estas notas, I es el isoespín, que nosotros denotamos por T)

from Coulomb interaction. On the other hand, the force in the $T = 0$ state can be different. For instance, the only two-nucleon bound state is the deuterium, an np state. What this suggests is that the bound state is in the $T = 0$ state, and anti-symmetric isospin wave function. Then the rest of the wave function must be symmetric. For a given potential, the S-wave is always more binding than the P-wave just because it lacks the centrifugal barrier. Therefore the deuterium is likely to be in the S-wave, a symmetric spatial wave function. Then the spin wave function must be symmetric, $S = 1$. Indeed deuterium does have spin one. A more quantitative test can be seen in Fig. 8. ^{21}F, ^{21}Ar, ^{21}Na, and ^{21}Mg all have the mass number 21. Assuming ^{18}F is in the $T = 0$ state, all four nuclei can be obtained by adding three ~~neutrons~~ nucleons to it, which can be in either $T = 3/2$ or $T = 1/2$ states. The nuclear excitation spectra show states common only between ^{21}Ar and ^{21}Na, which are in the $T = 1/2$ state, or states common to all four of them, which are in the $T = 3/2$ state. Similar check can be done among ^{14}C, ^{14}N, ^{14}O, which show states common to all of them ($T = 1$) or states special to ^{14}N ($T = 0$).

4 Yukawa Theory and Two-nucleon System

Given the properties of the nuclear force described in the previous section, what, after all, is it? I briefly go through the explanations in a quasi-historic way, but this is by no means rigorous or exhaustive. But hopefully I can give you an idea on how we came up with the current understanding, namely Quantum ChromoDynamics (QCD).

The obvious oddity with the nuclear force was its short-rangedness. People knew gravity and electromagnetism; both of them are long-ranged, with their potential decreasing as $1/r$. On the other hand, the nuclear force is practically zero beyond a few fm. As we will discuss in the "Quantization of Radiation Field," the electromagnetic interaction is described by photons in the fully quantum theory. Likewise, the nuclear force must also involve a particle that is responsible for the force. Such a particle is often called a "force carrier." The idea of the force carrier is simple: quantum mechanics allows you to "borrow" energy ΔE violating its conservation law as long as you give it back within time $\Delta t \sim \hbar/\Delta E$ allowed by the uncertainty principle. Take the case of an electromagnetic reaction, say electron proton scattering. An electron cannot emit a photon by itself because that would violate energy and momentum conservation. But it can do so by "borrowing" energy as

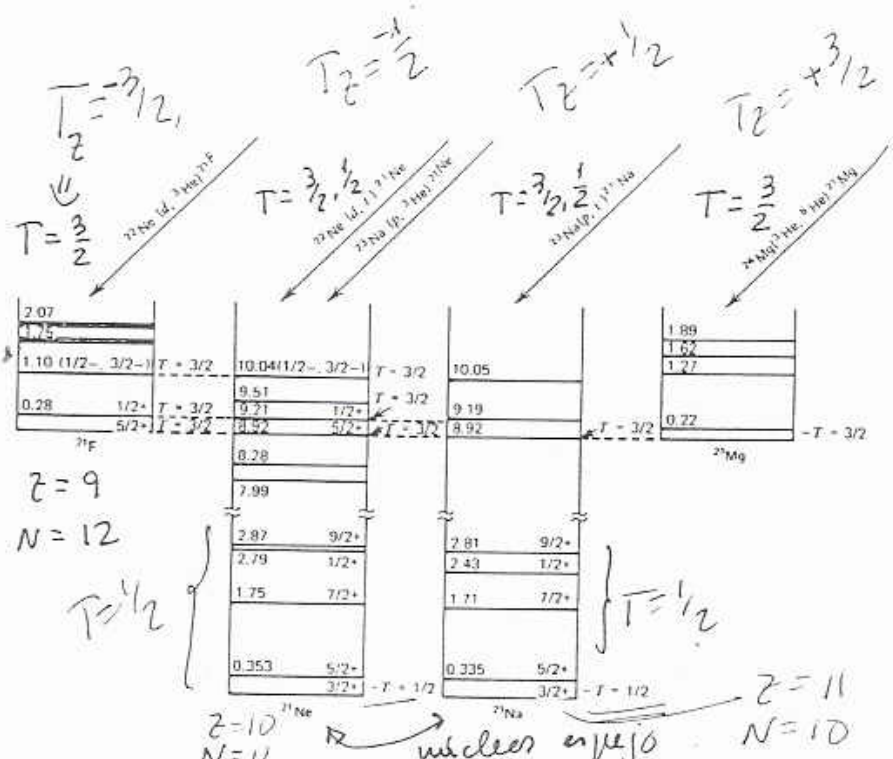

FIG. 6.1. Energy level diagrams for the members of the mass-21 isospin quartet showing the positions of the $T = 3/2$ levels in each nucleus. For clarity, the ground state energies of mirror-nuclei have been equated, and many of the excited levels of Ne and Na below 10-MeV excitation have been deleted [Butler, Cerny, and McCarthy, (68)].

Figure 8: Comparison of excitation spectrum of four nuclei with the same mass number, showing states with $T = 1/2$ and $T = 3/2$ multiplet structure. From "Theoretical Nuclear Physics," by Amos deShalit and Herman Feshbach, New York, Wiley, 1974.

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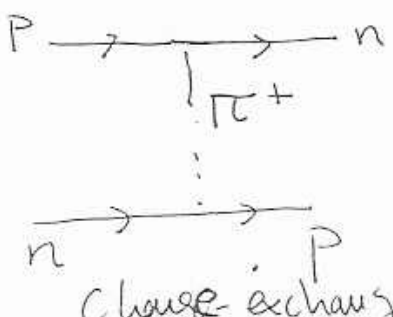
long as the created photon is absorbed by the proton within Δt allowed by the uncertainty principle. Then the "virtual photon" has propagated from the electron to the proton, causing a scattering process, because of its kick when emitted by the electron and when absorbed by the proton. Since the photon is a massless particle with $E = cp$, its energy can be arbitrarily small for small momenta, and hence Δt can be arbitrarily long. The distance the "virtual photon" can propagate can also be arbitrarily long $d = c\Delta t$. This is why the electromagnetic interaction is long-ranged. If, on the other hand, the force carrier had a finite mass m, there is a minimum energy required to create the force carrier particle $E_{min} = mc^2$. Therefore the time to pay back the debt is limited: $\Delta t = \hbar/mc^2$. The distance the force carrier can go within the allowed time limit is then also limited: $d = c\Delta t = \hbar/mc$. Therefore the force carrier cannot go beyond this distance and the force becomes short-ranged. This distance determined by the mass of the particle is called "Compton wavelength." Yukawa suggested back in 30's that the force carrier of the nuclear force must therefore be massive. Judging from the range of the nuclear force of about two fm, he suggested that the force carrier must weigh about 200 times electron, or 100 MeV/c^2. The short-rangedness is then an immediate consequence of the finite mass.

The presence of the charge exchange reaction suggests that the force carrier is (or at least can be) electrically charged. This particle is called charged pion π^- or π^+ in the modern terminology. The charge exchange reaction, producing the backward peak in the np scattering is caused by the following process. When the neutron comes close to the proton, the neutron emits the force carrier π^-, and it becomes a proton (!). Even though (from the neutron point of view) "she" is still going pretty much straight ahead, we see the proton coming along the original direction of the neutron, namely the "backscattered proton." The emitted π^- is then absorbed within the time allowed by the uncertainty principle and the proton becomes a neutron.

By 40's there was discovered a particle that weighs 200 times electron in cosmic rays (or more precisely, 105.7 MeV/c^2). This of course raised hope that the discovered particle may be the force carrier for the nuclear force. After intensive research, however, especially that carried out by Italians hiding (literally) underground in Rome under Nazi's occupation in 1945, it was shown that the new particle does not show any sign to feel the nuclear force. This particle is what is now called muon $\mu^\pm$. Indeed, underground is a good place to study muons! Later on people speculated that there may be *two* new particles weighing 200 times electron, and this is indeed what happened. By



Charge exchange con π^+

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$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot\vec{A}\times\vec{B}$

going to higher altitudes on the Andes in cosmic ray studies, people have found that the charged pions exist in cosmic rays, which quickly (within about 10^{-8} sec) decay to muons which live longer (about 10^{-6} sec) and reach the surface of the Earth. (Of course their life is stretched by the relativistic time dilation effect. Otherwise we didn't have a chance to detect them even on the Andes.) Only at higher altitudes, pions had chance to enter the detector (photographic films). Later on, a neutral pion π^0 was also discovered that decays into two photons. They are later determined to have no spin and odd parity. Once found, it seemed to confirm Yukawa's suggestion. The potential between nucleons caused by the exchange of a "virtual pion" was calculated to have the following form

$$V = \frac{1}{3}\frac{g^2}{\hbar c}\frac{m_\pi^2}{4m_N^2} m_\pi c^2 (\vec{\tau}_1\cdot\vec{\tau}_2)\left[(\vec{\sigma}_1\cdot\vec{\sigma}_2) + \left(1+\frac{3}{\mu r}+\frac{3}{(\mu r)^2}\right)S_{12}\right]\frac{e^{-\mu r}}{\mu r}. \quad (6)$$

Here $\mu = m_\pi c/\hbar$ with m_π with the small difference between $m_{\pi^\pm} = 139.6$ MeV/c^2 and $m_{\pi^0} = 135.0$ MeV ignored in the same spirit as we ignore the proton-neutron mass difference and call it m_N. The factor

$$S_{12} = \frac{1}{r^2}[3(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r}) - (\vec{\sigma}_1\cdot\vec{\sigma}_2)r^2] \quad (7)$$

is the form for the phenomenologically required tensor force. The matrices $\vec{\tau} = 2\vec{T}$ are the analogs of Pauli matrices for the isospin. The important point with the potential is that it is indeed invariant under the rotation of the isospin space because of the form $(\vec{\tau}_1\cdot\vec{\tau}_2)$.

The OPE (one-pion-exchange) exchange Eq. (6) works well in the two-nucleon system. We have seen that there is only one bound state in two-nucleon system, namely deuteron, with $T = 0$, $L = 0$, $S = 1$. Let us see if this is consistent with the OPE potential. We focus on the s-wave ($L = 0$) which doesn't have the centrifugal barrier and presumably binds the most. When $T = 1$ ($(\vec{\tau}_1\cdot\vec{\tau}_2) = +1$), the Fermi statistics requires $S = 0$ ($\vec{\sigma}_2 = -\vec{\sigma}_1$ and hence $(\vec{\sigma}_1\cdot\vec{\sigma}_2) = -3$). Then the tensor force is proportional to

$$S_{12} = \frac{1}{r^2}[3(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r}) - (\vec{\sigma}_1\cdot\vec{\sigma}_2)r^2] = \frac{1}{r^2}[-3(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_1\cdot\vec{r}) + 3r^2]. \quad (8)$$

At the lowest order in the potential in perturbation theory, using the fact that the s-wave is isotropic, we find $\langle r^i r^j\rangle = \frac{1}{3}\langle r^2\rangle$, and hence the tensor force vanishes identically. Therefore, the OPE potential is

$$V = -\frac{g^2}{\hbar c}\frac{m_\pi^2}{4m_N^2} m_\pi c^2 \frac{e^{-\mu r}}{\mu r}. \quad (9)$$

τ

$\vec{T} = \vec{t}_1 + \vec{t}_2$, $T^2 = t_1^2 + t_2^2 + 2\vec{t}_1\cdot\vec{t}_2 \Rightarrow$

$\Rightarrow \vec{\tau}_1\cdot\vec{\tau}_2 = \dfrac{T^2 - \tau_1^2 - \tau_2^2}{2}$

pero $\tau_1^2 = 4$

$\tau^2 = 4$

This potential is attractive, of finite range, and may or may not have a bound state depending on the size of the coupling $g^2/\hbar c$ and m_π. For the actual values, there is no bound state.

On the other hand, for the $T = 0$, $L = 0$, $S = 1$ case, we have $(\vec{\tau}_1 \cdot \vec{\tau}_2) = -3$ and $(\vec{\sigma}_1 \cdot \vec{\sigma}_2) = +1$. Let us take $S_z = +1$ state as an example. Then the tensor force does not vanish, and its expectation value is proportional to

$$\begin{aligned}
&\langle S = 1, S_z = +1 | \frac{1}{r^2}[3(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r}) - (\vec{\sigma}_1 \cdot \vec{\sigma}_2) r^2] | S = 1, S_z = +1 \rangle \\
&= \langle S = 1, S_z = +1 | \frac{1}{r^2}[3(\sigma_1^z z)(\sigma_2^z z) - r^2] | S = 1, S_z = +1 \rangle \\
&= \frac{2z^2 - x^2 - y^2}{r^2}. \qquad (10)
\end{aligned}$$

Therefore, the OPE potential is

$$V = -\frac{g^2}{\hbar c} \frac{m_\pi^2}{4m_N^2} m_\pi c^2 \left[1 + \frac{2z^2 - x^2 - y^2}{r^2} \right] \frac{e^{-\mu r}}{\mu r}. \qquad (11)$$

The coefficient of the potential is the same as the $T = 1$ case, except that there is an addition of the quadrupole moment $r^2 Y_2^0 = \sqrt{\frac{5}{16\pi}}(2z^2 - x^2 - y^2)$. If the quadrupole moment is positive, which means a cigar-like shape, as opposed to negative, which means a pancake like shape, the quadrupole moment adds to the attractive force and can lead to a bound state even if the $T = 1$ case doesn't. Experimentally, the quadrupole moment of the deuteron is confirmed and has the value $Q(d) = 2.78 \times 10^{-27}$ cm^2. The deuteron indeed has a cigar-like shape where the spins are lined up along the elongated direction.

In order for the quadrupole moment to be non-vanishing, however, a pure S-wave would not do the job because it is completely isotropic. However, the state with total $J = 1$ with $S = 1$ can also arise from $L = 2$. In fact, the deteron has a mixture of $L = 2$ state that is responsible for the finite quadrupole moment.

5 Fundamental Description of Nuclear Force

Now the world looked simple: there are protons and neutrons in nuclei, bound together by the force mediated by the exchange of pions. But the

With $a_c = 0.72$ MeV and $a_{sym} = 23$ MeV, it follows that the first two terms in the numerator are negligible, and so

$$Z_{min} \simeq \frac{A}{2\left(1 + \frac{1}{4}A^{2/3}a_c/a_{sym}\right)} \tag{3.31}$$

For small A, $Z_{min} = A/2$ as expected, but for large A, $Z_{min} < A/2$. For heavy nuclei, Equation 3.31 gives $Z/A = 0.41$, consistent with observed values for heavy stable nuclei.

Figure 3.18 shows a typical odd-A decay chain for $A = 125$, leading to the stable nucleus at $Z = 52$. The unstable nuclei approach stability by converting a neutron into a proton or a proton into a neutron by radioactive β decay. Notice how the decay energy (that is, the mass difference between neighboring isobars) increases as we go further from stability. For even A, the pairing term gives two parabolas, displaced by 2δ. This permits two unusual effects, not seen in odd-A decays: (1) some odd-Z, odd-N nuclei can decay in either direction, converting a neutron to a proton or a proton to a neutron; (2) certain *double β decays* can become energetically possible, in which the decay may change 2 protons to 2 neutrons. Both of these effects are discussed in Chapter 9.

3.4 NUCLEAR ANGULAR MOMENTUM AND PARITY

In Section 2.5 we discussed the coupling of orbital angular momentum ℓ and spin s to give total angular momentum j. To the extent that the nuclear potential is central, ℓ and s (and therefore j) will be constants of the motion. In the quantum mechanical sense, we can therefore label every nucleon with the corresponding quantum numbers ℓ, s, and j. The total angular momentum of a nucleus containing A nucleons would then be the vector sum of the angular momenta of all the nucleons. This total angular momentum is usually called the *nuclear spin* and is represented by the symbol I. The angular momentum I has all of the usual properties of quantum mechanical angular momentum vectors: $I^2 = \hbar^2 I(I+1)$ and $I_z = m\hbar$ $(m = -I, \ldots, +I)$. For many applications involving angular momentum, the nucleus behaves as if it were a single entity with an intrinsic angular momentum of I. In ordinary magnetic fields, for example, we can observe the nuclear Zeeman effect, as the state I splits up into its $2I + 1$ individual substates $m = -I, -I + 1, \ldots, I - 1, I$. These substates are equally spaced, as in the atomic normal Zeeman effect. If we could apply an incredibly strong magnetic field, so strong that the coupling between the nucleons were broken, we would see each individual j splitting into its $2j + 1$ substates. Atomic physics also has an analogy here: when we apply large magnetic fields we can break the coupling between the electronic ℓ and s and separate the $2\ell + 1$ components of ℓ and the $2s + 1$ components of s. No fields of sufficient strength to break the coupling of the nucleons can be produced. We therefore observe the behavior of I as if the nucleus were only a single "spinning" particle. For this reason, the spin (total angular momentum) I and the corresponding spin quantum number I are used to describe nuclear states.

To avoid confusion, we will always use I to denote the nuclear spin; we will use j to represent the total angular momentum of a single nucleon. It will often

be the case that a single valence particle determines all of the nuclear properties; in that case, $I = j$. In other cases, it may be necessary to consider two valence particles, in which case $\boldsymbol{I} = \boldsymbol{j}_1 + \boldsymbol{j}_2$, and several different resultant values of I may be possible. Sometimes the odd particle and the remaining core of nucleons each contribute to the angular momentum, with $\boldsymbol{I} = \boldsymbol{j}_{\text{particle}} + \boldsymbol{j}_{\text{core}}$.

One important restriction on the allowed values of I comes from considering the possible z components of the total angular momentum of the individual nucleons. Each j must be half-integral ($\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$) and thus its only possible z components are likewise half-integral ($\pm\frac{1}{2}\hbar, \pm\frac{3}{2}\hbar, \pm\frac{5}{2}\hbar, \ldots$). If we have an even number of nucleons, there will be an even number of half-integral components, with the result that the z component of the total I can take only integral values. This requires that I itself be an integer. If the number of nucleons is odd, the total z component must be half-integral and so must the total I. We therefore require the following rules:

odd-A nuclei: I = half-integral

even-A nuclei: I = integral

The measured values of the nuclear spin can tell us a great deal about the nuclear structure. For example, of the hundreds of known (stable and radioactive) even-Z, even-N nuclei, all have spin-0 ground states. This is evidence for the nuclear pairing force we discussed in the previous section; the nucleons couple together in spin-0 pairs, giving a total I of zero. As a corollary, the ground state spin of an odd-A nucleus must be equal to the j of the odd proton or neutron. We discuss this point further when we consider the nuclear shell model in Chapter 5.

Along with the nuclear spin, the *parity* is also used to label nuclear states. The parity can take either + (even) or − (odd) values. If we knew the wave function of every nucleon, we could determine the nuclear parity by multiplying together the parities of each of the A nucleons, ending with a result π either + or −: $\pi = \pi_1\pi_2 \cdots \pi_A$. However, in practice no such procedure is possible, for we generally cannot assign a definite wave function of known parity to every nucleon. Like the spin I, we regard the parity π as an "overall" property of the whole nucleus. It can be directly measured using a variety of techniques of nuclear decays and reactions. The parity is denoted by a + or − superscript to the nuclear spin, as I^{π}. Examples are 0^+, 2^-, $\frac{1}{2}^-$, $\frac{5}{2}^+$. There is no direct theoretical relationship between I and π; for any value of I, it is possible to have either $\pi = +$ or $\pi = -$.

3.5 NUCLEAR ELECTROMAGNETIC MOMENTS

Much of what we know about nuclear structure comes from studying not the strong nuclear interaction of nuclei with their surroundings, but instead the much weaker electromagnetic interaction. That is, the strong nuclear interaction establishes the distribution and motion of nucleons in the nucleus, and we probe that distribution with the electromagnetic interaction. In doing so, we can use electromagnetic fields that have less effect on the motion of nucleons than the

where $|\vec{\ell}|$ is the classical angular momentum mvr. In quantum mechanics, we operationally *define* the observable magnetic moment to correspond to the direction of greatest component of $\vec{\ell}$; thus we can take Equation 3.32 directly into the quantum regime by replacing $\vec{\ell}$ with the expectation value relative to the axis where it has maximum projection, which is $m_\ell h$ with $m_\ell = +\ell$. Thus

$$\mu = \frac{eh}{2m}\ell \tag{3.33}$$

where now ℓ is the angular momentum quantum number of the orbit.

The quantity $eh/2m$ is called a *magneton*. For atomic motion we use the electron mass and obtain the *Bohr magneton* $\mu_B = 5.7884 \times 10^{-5}$ eV/T. Putting in the proton mass we have the *nuclear magneton* $\mu_N = 3.1525 \times 10^{-8}$ eV/T. Note that $\mu_N \ll \mu_B$ owing to the difference in the masses; thus under most circumstances atomic magnetism has much larger effects than nuclear magnetism. Ordinary magnetic interactions of matter (ferromagnetism, for instance) are determined by atomic magnetism; only in very special circumstances can we observe the effects of nuclear magnetism (see Chapter 16).

We can rewrite Equation 3.33 in a more useful form:

$$\mu = g_\ell \ell \mu_N \tag{3.34}$$

where g_ℓ is the *g factor* associated with the orbital angular momentum ℓ. For protons $g_\ell = 1$; because neutrons have no electric charge, we can use Equation 3.34 to describe the orbital motion of neutrons if we put $g_\ell = 0$.

We have thus far been considering only the orbital motion of nucleons. Protons and neutrons, like electrons, also have intrinsic or spin magnetic moments, which have no classical analog but which we write in the same form as Equation 3.34:

$$\mu = g_s s \mu_N \tag{3.35}$$

where $s = \frac{1}{2}$ for protons, neutrons, and electrons. The quantity g_s is known as the *spin g factor* and is calculated by solving a relativistic quantum mechanical equation. For a spin-$\frac{1}{2}$ point particle such as the electron, the Dirac equation gives $g_s = 2$, and measurement is quite consistent with that value for the electron: $g_s = 2.0023$. The difference between g_s and 2 is quite small and can be very accurately computed using the higher order corrections of quantum electrodynamics. On the other hand, for free nucleons, the experimental values are far from the expected value for point particles:

$$\text{proton:} \quad g_s = 5.5856912 \pm 0.0000022$$
$$\text{neutron:} \quad g_s = -3.8260837 \pm 0.0000018$$

(The measured magnetic moments, in nuclear magnetons, are just half the g_s factors.) Not only is the proton value far from the expected value of 2 for a point particle, but the uncharged neutron has a nonzero magnetic moment! Here is perhaps our first evidence that the nucleons are not elementary point particles like the electron, but have an internal structure; the internal structure of the nucleons must be due to charged particles in motion, whose resulting currents give the observed spin magnetic moments. It is interesting to note that g_s for the proton is greater than its expected value by about 3.6, while g_s for the neutron is

strong force of the nuclear environment; thus our measurements do not seriously distort the object we are trying to measure.

Any distribution of electric charges and currents produces electric and magnetic fields that vary with distance in a characteristic fashion. It is customary to assign to the charge and current distribution an electromagnetic *multipole moment* associated with each characteristic spatial dependence—the $1/r^2$ electric field arises from the net charge, which we can assign as the zeroth or *monopole* moment; the $1/r^3$ electric field arises from the first or *dipole* moment; the $1/r^4$ electric field arises from the second or *quadrupole* moment, and so on. The magnetic multipole moments behave similarly, with the exception of the monopole moment; as far as we know, magnetic monopoles either do not exist or are exceedingly rare, and thus the magnetic monopole field ($\propto 1/r^2$) does not contribute. Electromagnetic theory gives us a recipe for calculating the various electric and magnetic multipole moments, and the same recipe can be carried over into the nuclear regime using quantum mechanics, by treating the multipole moments in operator form and calculating their expectation values for various nuclear states. These expectation values can then be directly compared with the experimental values we measure in the laboratory. Techniques for measuring the nuclear moments are discussed in Chapter 16.

The simplest distributions of charges and currents give only the lowest order multipole fields. A spherical charge distribution gives only a monopole (Coulomb) field; the higher order fields all vanish. A circular current loop gives only a magnetic dipole field. Nature has not been arbitrary in the construction of nuclei; if a simple, symmetric structure (consistent with the nuclear interaction) is possible, then nuclei tend to acquire that structure. It is therefore usually necessary to measure or calculate only the lowest order multipole moments to characterize the electromagnetic properties of the nucleus.

Another restriction on the multipole moments comes about from the symmetry of the nucleus, and is directly related to the parity of the nuclear states. Each electromagnetic multipole moment has a parity, determined by the behavior of the multipole operator when $\boldsymbol{r} \rightarrow -\boldsymbol{r}$. The parity of electric moments is $(-1)^L$, where L is the order of the moment ($L = 0$ for monopole, $L = 1$ for dipole, $L = 2$ for quadrupole, etc.); for magnetic moments the parity is $(-1)^{L+1}$. When we compute the expectation value of a moment, we must evaluate an integral of the form $\int \psi^* \mathcal{O} \psi \, dv$, where $\mathcal{O}$ is the appropriate electromagnetic operator. The parity of ψ itself is not important; because ψ appears twice in the integral, whether $\psi \rightarrow +\psi$ or $\psi \rightarrow -\psi$ does not change the integrand. If, however, $\mathcal{O}$ has odd parity, then the integrand is an odd function of the coordinates and must vanish identically. *Thus all odd-parity static multipole moments must vanish*—electric dipole, magnetic quadrupole, electric octupole ($L = 3$), and so on.

The monopole electric moment is just the net nuclear charge Ze. The next nonvanishing moment is the *magnetic dipole moment* μ. A circular loop carrying current i and enclosing area A has a magnetic moment of magnitude $|\mu| = iA$; if the current is caused by a charge e, moving with speed v in a circle of radius r (with period $2\pi r/v$), then

$$|\mu| = \frac{e}{(2\pi r/v)} \pi r^2 = \frac{evr}{2} = \frac{e}{2m} |\ell| \tag{3.32}$$

Table 3.2 Sample Values of Nuclear Magnetic Dipole Moments

Nuclide	$\mu(\mu_N)$
n	−1.9130418
p	+2.7928456
^{2}H (D)	+0.8574376
^{17}O	−1.89379
^{57}Fe	+0.09062293
^{57}Co	+4.733
^{93}Nb	+6.1705

All values refer to the nuclear ground states; uncertainties are typically a few parts in the last digit. For a complete tabulation, see V. S. Shirley, in *Table of Isotopes* (Wiley: New York, 1978), Appendix VII.

less than its expected value (zero) by roughly the same amount. Formerly these differences between the expected and measured g_s values were ascribed to the clouds of π mesons that surround nucleons, with positive and neutral π mesons in the proton's cloud, and negative and neutral π mesons in the neutron's cloud. The equal and opposite contributions of the meson cloud are therefore not surprising. In present theories we consider the nucleons as composed of three quarks; adding the magnetic moments of the quarks gives the nucleon magnetic moments directly (see Chapter 18).

In nuclei, the pairing force favors the coupling of nucleons so that their orbital angular momentum and spin angular momentum each add to zero. Thus the paired nucleons do not contribute to the magnetic moment, and we need only consider a few valence nucleons. If this were not so, we might expect on statistical grounds alone to see a few heavy nuclei with very large magnetic moments, perhaps tens of nuclear magnetons. However, no nucleus has been observed with a magnetic dipole moment larger than about $6\mu_N$.

Table 3.2 gives some representative values of nuclear magnetic dipole moments. Because of the pairing force, we can analyze these magnetic moments to learn about the nuclear structure. In Chapter 4, we discuss the magnetic moment of the deuteron, and in Chapter 5 we consider how nuclear models predict the magnetic moments of heavier nuclei.

The next nonvanishing moment is the *electric quadrupole moment*. The quadrupole moment eQ of a classical point charge e is of the form $e(3z^2 - r^2)$. If the particle moves with spherical symmetry, then (on the average) $z^2 = x^2 = y^2 = r^2/3$ and the quadrupole moment vanishes. If the particle moves in a classical flat orbit, say in the xy plane, then $z = 0$ and $Q = -r^2$. The quadrupole moment in quantum mechanics is

$$eQ = e\int \psi^*(3z^2 - r^2)\psi \, d\tau \tag{3.36}$$

for a single proton; for an orbiting neutron, $Q = 0$. If $|\psi|^2$ is spherically symmetric, then $Q = 0$. If $|\psi|^2$ is concentrated in the xy plane ($z \cong 0$), then

Table 3.3 Some Values of Nuclear Electric Quadrupole Moments

Nuclide	Q (b)
^{2}H (D)	+0.00288
^{17}O	−0.02578
^{59}Co	+0.40
^{63}Cu	−0.209
^{133}Cs	−0.003
^{161}Dy	+2.4
^{176}Lu	+8.0
^{209}Bi	−0.37

All values refer to nuclear ground states; uncertainties are typically a few parts in the last digit. For a complete tabulation, see V. S. Shirley, in *Table of Isotopes* (Wiley: New York, 1978), Appendix VII.

$Q \sim -\langle r^2 \rangle$, while if $|\psi|^2$ is concentrated along the z axis ($z \cong r$), we might have $Q \sim +2\langle r^2 \rangle$. Here $\langle r^2 \rangle$ is the mean-square radius of the orbit. Once again the pairing force is helpful, for if the paired nucleons move in spherically symmetric orbits, they do not contribute to Q. We might therefore expect that for many nuclei, the quadrupole moment can be estimated from the valence nucleon, which we can assume to orbit near the surface, so $r = R_0 A^{1/3}$. We therefore estimate $|eQ| \leq eR_0^2 A^{2/3}$, which ranges from about 6×10^{-30} cm^2 for light nuclei to 50×10^{-30} cm^2 for heavy nuclei. The unit of 10^{-28} m^2 is used frequently in nuclear reaction studies for cross sections, and is known as a *barn* (b). This unit is also convenient for measuring quadrupole moments; thus the expected maximum is from 0.06 to 0.5 eb. As you can see from Table 3.3, many nuclei do fall within that range, but several, especially in the rare-earth region, are far outside. Here the quadrupole moment is giving important information—the model of the single particle cannot explain the large observed quadrupole moments. Most or all of the protons must somehow collectively contribute to have such a large Q. The assumption of a spherically symmetric core of paired nucleons is not valid for these nuclei. The core in certain nuclei can take on a static nonspherical shape that can give a large quadrupole moment. The properties of such strongly deformed nuclei are discussed in Chapter 5.

3.6 NUCLEAR EXCITED STATES

Just as we learn about atoms by studying their excited states, we study nuclear structure in part through the properties of nuclear excited states. (And like atomic excited states, the nuclear excited states are unstable and decay rapidly to the ground state.) In atoms, we make excited states by moving individual electrons to higher energy orbits, and we can do the same for individual nucleons; thus the excited states can reveal something about the orbits of individual nucleons. We have already several times in this chapter referred to the complementary single-particle and collective structure of nuclei—we can also

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NUCLEAR SPIN AND MOMENTS

In analyzing and interpreting nuclear level schemes in this text, we have repeatedly used the spin quantum number I to label the individual levels. In Chapter 5 we discussed how important it is to have an established set of these spin quantum numbers to compare the observed level scheme with the predictions of a particular nuclear model. The measurement of nuclear spin assignments is one of the goals of experimental nuclear physics, and in this chapter we explore some of the techniques that are used to obtain this information.

Nuclear magnetic dipole and electric quadrupole moments have a similar importance in helping us to interpret nuclear structure. We have already discussed in Chapter 4 the clues to the deuteron structure deduced from its moments. In Chapter 5 we have seen the systematic behavior of shell-model magnetic moments, and we have also seen how the unusually large quadrupole moments of certain nuclei suggest a new feature of nuclear structure, the stable deformation.

The experimental techniques that are responsible for the determination of these spins and moments span a considerable range, from those involving nuclear radiations (angular distributions and correlations, Mössbauer effect), to those involving atomic and molecular beams (the Stern-Gerlach experiment, for instance) and radiations in the optical, microwave, and radio regions of the spectrum. In this chapter we introduce and review many of these techniques and give examples of their applications.

16.1 NUCLEAR SPIN

Each nuclear state is assigned a unique "spin" quantum number I, representing the total angular momentum (orbital plus intrinsic) of all the nucleons in the nucleus. The vector I can be considered the sum of the orbital and intrinsic

contributions to the angular momentum:

$$\boldsymbol{I} = \sum_{i=1}^{A} (\boldsymbol{\ell}_i + \boldsymbol{s}_i) \tag{16.1}$$

$$= \boldsymbol{L} + \boldsymbol{S} \tag{16.2a}$$

$$= \sum_{i=1}^{A} \boldsymbol{j}_i \tag{16.2b}$$

where the decomposition according to either Equation 16.2*a* or 16.2*b* is largely a matter of convenience. The *quantum number* I has the usual connection with the *vector* $\boldsymbol{I}$:

$$|\boldsymbol{I}| = \sqrt{I(I+1)}\,\hbar \tag{16.3}$$

$$I_z = m_I \hbar \qquad (m_I = I, I-1, \ldots, -I+1, -I) \tag{16.4}$$

Equation 16.1 represents what could in principle be a very complicated coupling of many vectors to a single resultant, and it may not be apparent why we can neglect this internal structure and treat the nucleus as if it were an elementary particle with a single spin quantum number, representing the intrinsic angular momentum of the "particle." This is possible only because the interactions to which we subject the nucleus, such as static electromagnetic fields, are not sufficiently strong to change the internal structure or break the coupling of nucleons that is responsible for Equation 16.1.

For the electronic motion in atoms, we can similarly define the total electronic angular momentum:

$$\boldsymbol{J} = \sum_{i=1}^{Z} \left(\boldsymbol{\ell}_i^{(e)} + \boldsymbol{s}_i^{(e)}\right) \tag{16.5}$$

where the ℓ and s vectors now refer to the electronic states. In analogy with the nuclear case, we can often (but not always) treat the electrons as if they were represented by a single angular momentum $\boldsymbol{J}$.

Finally, there are cases in which it is most appropriate to deal with the total nuclear plus electronic angular momentum, usually called $\boldsymbol{F}$:

$$\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{J} \tag{16.6}$$

The vectors $\boldsymbol{J}$ and $\boldsymbol{F}$ obey all the usual quantum rules for angular momentum, as in Equations 16.3 and 16.4.

The quantum numbers I and J may be either integral or half-integral as the number of nucleons or electrons is even or odd:

A	Z	I	J	F
Even	Even	Integer	Integer	Integer
Odd	Even	Half-integer	Integer	Half-integer
Even	Odd	Integer	Half-integer	Half-integer
Odd	Odd	Half-integer	Half-integer	Integer

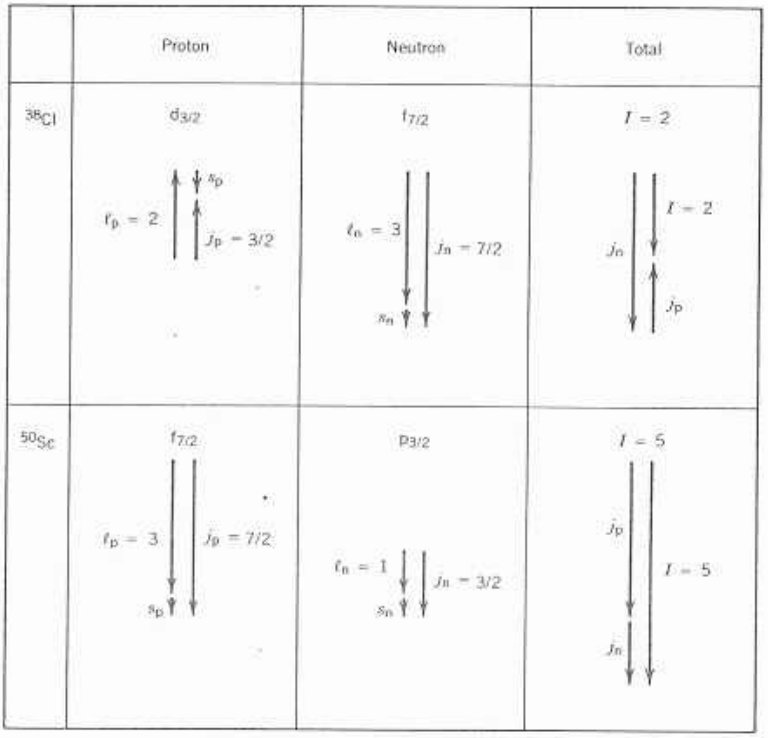

Figure 16.1 Proton-neutron angular momentum couplings in ^{38}Cl and ^{50}Sc.

For nuclear ground states, there are several rules for determining the spins:

1. All even-Z, even-N nuclei have $I = 0$. This results from the strong tendency of nucleons to couple pairwise to zero spin.
2. In odd-A nuclei, the net spin is almost always determined by the j of the last odd particle, with the remaining $A - 1$ nucleons (having even numbers of protons and neutrons) pairing to zero spin as above.
3. In odd-Z, odd-N nuclei, the spin is determined by the vector coupling of the j of the odd proton and neutron, $\boldsymbol{I} = \boldsymbol{j}_p + \boldsymbol{j}_n$, and thus any of several values are possible. To determine which of these possible couplings will be the ground state, we use the empirical rule that the ground state is usually the coupling with the neutron and proton intrinsic spins $\boldsymbol{s}_p$ and $\boldsymbol{s}_n$ parallel. As an example, consider ^{38}Cl, which consists of a $d_{3/2}$ proton coupled to an $f_{7/2}$ neutron. For the proton, $\ell_p = 2$ and thus $\boldsymbol{s}_p$ is opposite to $\boldsymbol{j}_p$. For the neutron, $\ell_n = 3$ and $\boldsymbol{s}_n$ is parallel to $\boldsymbol{j}_n$. Arranging the coupling so that $\boldsymbol{s}_p$ and $\boldsymbol{s}_n$ are parallel, as in Figure 16.1, we get $I = |j_p - j_n|$ or $I = 2$, which is in fact the ground-state spin of ^{38}Cl. (The first excited state is $I = 5$, corresponding to $I = j_p + j_n$.) On the other hand, consider ^{50}Sc, resulting from an $f_{7/2}$ proton coupled to a $p_{3/2}$ neutron. Here making $\boldsymbol{s}_p$ and $\boldsymbol{s}_n$

parallel also makes j_p and j_n parallel, and thus $I = j_p + j_n = 5$, in agreement with observation. (The state $I = |j_p - j_n| = 2$ is a low excited state of ^{50}Sc.) Other couplings with I between $j_p + j_n$ and $|j_p - j_n|$ may be found among other low-lying excited states.

16.2 NUCLEAR MOMENTS

Magnetic Dipole Moments

Classically, the magnetic dipole moment μ arises from the motion of charged particles, and we can regard μ as a means to characterize a distribution of currents whose effect on the surroundings (that is, on other moving charges) we call "magnetic." When we go over to the quantum limit we find a similar relationship, with one distinctly nonclassical addition: the intrinsic angular momentum (spin) contributes to the magnetic moment also.

Let's briefly review the classical electromagnetism that leads to magnetic dipole moments. We consider some currents distributed over a sample that occupies a certain volume in space (Figure 16.2). The distribution of currents is specified by the *current density* $j(r')$. The vector r' locates a specific point of the sample relative to the origin; the *vector function* $j(r')$ then gives the magnitude and direction of the electric current per unit volume dv' at that point. The recipe for calculating the magnetic field B resulting from the currents is straightforward: first calculate the vector potential $A(r)$ at the observation point r by integrating (summing) over all the currents in the sample:

$$A(r) = \frac{\mu_0}{4\pi} \int \frac{j(r')\, dv'}{|r - r'|} \tag{16.7}$$

and then the magnetic field follows directly from $B(r) = \nabla \times A(r)$. Following some mathematical manipulations, which can be found in standard texts on

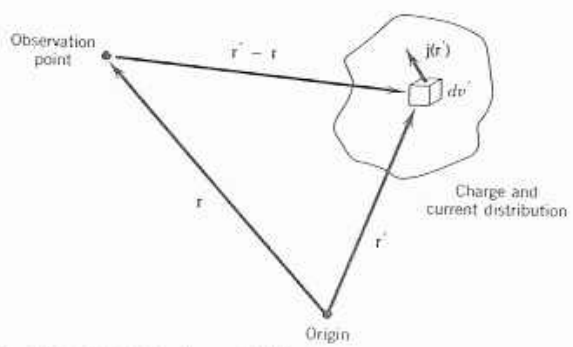

Figure 16.2 The current element $j(r')dv'$ gives a contribution to the vector potential at the observation point. The total potential is found from the integral over the entire current distribution.

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electromagnetism, we can rewrite the vector potential as

$$A(r) = \frac{\mu_0}{4\pi}\left\{\frac{1}{r}\int j(r')\,dv' + \frac{1}{r^3}\int j(r')(r\cdot r')\,dv' + \cdots\right\} \tag{16.8}$$

which can be written

$$A(r) = \frac{\mu_0}{4\pi}\frac{\mu \times r}{r^3} + \cdots \tag{16.9}$$

where

$$\mu = \frac{1}{2}\int r' \times j(r')\,dv' \tag{16.10}$$

The leading nonvanishing term is characterized by the *magnetic dipole moment* μ of the current distribution. What we have done, in effect, is a multipole expansion of the current distribution; the lowest-order term (dipole) is likely to be the most important. The argument of the integral for μ includes the charge density and the vector product $r' \times v'$, which in the case of a particle with mass m is just ℓ/m, where ℓ is the angular momentum. Going over to the quantum limit, the charge density is $e|\psi(r')|^2$, and it is entirely consistent with our previous experience with quantum mechanics to write this as

$$\mu = \frac{e}{2m}\int \psi^*(r')\ell\psi(r')\,dv' \tag{16.11}$$

If the wave function corresponds to a state of definite ℓ_z, then only the z component of the integral is nonvanishing, and

$$\mu_z = \frac{e}{2m}\int \psi^*(r')\ell_z\psi(r')\,dv' \tag{16.12}$$

$$\mu_z = \frac{eh}{2m}m_\ell \tag{16.13}$$

with $\ell_z = m_\ell h$.

What we observe in an experiment as *the* magnetic moment is *defined* to be the value of μ_z corresponding to the maximum possible value of the z component of the angular momentum. The quantum number m_ℓ has a maximum value of $+\ell$, and thus the magnetic moment μ is

$$\mu = \frac{eh}{2m}\ell \tag{16.14}$$

The quantity $eh/2m$ has the dimensions of a magnetic moment (ℓ is a dimensionless quantum number) and is called a *magneton*. Putting in the proton mass for m, we get a *nuclear magneton* μ_N:

$$\mu_N = \frac{eh}{2m_p} = 3.15245 \times 10^{-8}\ \text{eV/T}$$

and using the electron mass gives the *Bohr magneton* μ_B:

$$\mu_B = \frac{eh}{2m_e} = 5.78838 \times 10^{-5}\ \text{eV/T}$$

Considering the intrinsic spin, which has no classical analog, we make a simple extension of Equation 16.14:

$$\mu = (g_\ell\ell + g_s s)\mu_N/\hbar \tag{16.15}$$

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where the *g factors* g_ℓ and g_s account for the orbital and intrinsic contributions to μ. Their values can be adjusted as needed for individual particles: $g_\ell = 1$ for protons and g_s must be measured for "free protons" in which ℓ does not contribute to μ. As we discuss later in this chapter, g_s is measured to be 5.5856912 for protons. For neutrons, which are uncharged, we can set $g_\ell = 0$, and g_s is measured to be -3.8260837.

In real nuclei, we must make a modification to allow for the effects of all the nucleons:

$$\mu = \sum_{i=1}^{A} [g_{\ell,i}\ell_i + g_{s,i}s_i]\mu_N/h \tag{16.16}$$

which is similar to Equation 16.1 for I.

There is no single theory that allows us to evaluate Equation 16.16 to calculate μ because the interactions between the nucleons are strong and the relative spin orientations are not sufficiently well known. In certain cases, we can make simplifying assumptions, based on nuclear models. For example, in the independent particle shell model, we couple $A - 1$ nucleons pairwise to zero-spin combinations that do not contribute to μ. For the remaining odd nucleon, the shell-model theory gives the coupling of ℓ and s to form I, which permits μ to be calculated, as we did in Section 5.1. In many other cases, we cannot ignore the effect of the "core" nucleons, and we assign them a "collective" g factor usually designated g_R, so that

$$\mu = \left[g_R I_c + \sum_i (g_{\ell,i}\ell_i + g_{s,i}s_i)\right]\mu_N/h \tag{16.17}$$

where I_c refers to the core and the sum is carried out over a few nucleons outside the core. If we consider "pure" collective states, with no odd nucleons, the collective model gives $g_R = Z/A$, the ratio of the nuclear charge to its mass. Figure 5.16*a* showed that this was a good approximation for 2^+ states of many even-Z, even-N nuclei.

Electric Quadrupole Moments

We now consider the distribution of charges, rather than currents, within the nucleus. From an external point, the electric potential $V(r)$ appears to be

$$V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')\,dv'}{|r - r'|} \tag{16.18}$$

which is analogous to the expression (16.7) for the magnetic vector potential. Classically, we can assign to a charge distribution a monopole (Coulomb) field, which is proportional to the total charge. If we construct a charge distribution in which the total charge vanishes, we can easily study the next highest multipole, the dipole field, the standard for which is charges of $\pm q$ located at, respectively, $z = +a/2$ and $z = -a/2$. In general, any charge distribution that lacks spherical symmetry will have a dipole field, possibly in addition to the monopole field. (One way to distinguish the two contributions to the total field is that the monopole electric field varies as r^{-2} while the dipole field varies as r^{-3}.) Just as

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adding equal and opposite charges at different locations gives a dipole field, adding equal and opposite dipoles causes the vanishing of the dipole field and gives the next higher multipole, the quadrupole field. For example, we could add the dipole with charges $-q$ at the origin and $+q$ at $z = a$ to the opposite dipole with charges $-q$ at the origin and $+q$ at $z = -a$. The characteristic dependence of the electric quadrupole field is as r^{-4}.

Expanding the factor of $|\boldsymbol{r} - \boldsymbol{r}'|$ in Equation 16.18 gives immediately the mathematical details of the multipole expansion of the electric field:

$$|\boldsymbol{r} - \boldsymbol{r}'|^{-1} = r^{-1}\left[1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta\right]^{-1/2} \tag{16.19}$$

$$\equiv \frac{1}{r}\left(1 - \frac{1}{2}\left(\frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta\right) + \frac{3}{8}\left(\frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta\right)^2 + \cdots\right) \tag{16.20}$$

where θ is the angle between r and r', and where we have assumed $r \gg r'$. (That is, the observation point is far from the nucleus. For the interaction with atomic electrons, which dominates the hyperfine structure, this is a good approximation.) Thus

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int\rho(\boldsymbol{r}')\,d\upsilon' + \frac{1}{r^2}\int\rho(\boldsymbol{r}')r'\cos\theta\,d\upsilon' + \frac{1}{r^3}\int\rho(\boldsymbol{r}')r'^2\tfrac{1}{2}(3\cos^2\theta - 1)\,d\upsilon' + \cdots\right] \tag{16.21}$$

The integral in the first term gives the total charge Ze, which from the point of view of nuclear structure is uninteresting. The second term vanishes for nuclei under ordinary circumstances because nuclear states are, to a very good approximation of the order of one part in 10^7, states of definite parity. Going over to the quantum limit and replacing $\rho(\boldsymbol{r}')$ by $\psi^*(\boldsymbol{r}')\psi(\boldsymbol{r}')$, the integral vanishes because the integrand is an odd function of the coordinates. (Simplify the geometry somewhat by choosing the origin at the center of the nuclear charge distribution, and let $\boldsymbol{r}$ define the z axis. Then $r'\cos\theta$ is z', and under the parity operation $z' \to -z'$ while $|\psi(\boldsymbol{r}')|^2 = |\psi(-\boldsymbol{r}')|^2$. The integrand is therefore odd and the integral vanishes.) The first "interesting" term in the multipole expansion is the quadrupole term, and we define *the* nuclear quadrupole moment as

$$eQ = \int\rho(\boldsymbol{r}')r'^2(3\cos^2\theta' - 1)\,d\upsilon' \tag{16.22}$$

where, as in the case of the magnetic dipole moment, we refer to a specific choice of reference axis—we measure θ' from the axis corresponding to the maximum projection of the nuclear spin.

The nuclear quadrupole moment tells us whether nuclei are spherical (for which $Q = 0$) or nonspherical. If $Q > 0$, the nuclei are *prolate* deformed—in the expression (16.22), the quantity $r'^2(3\cos^2\theta' - 1) = 3z'^2 - r'^2$ is on the average positive. That is, there is more of the nuclear charge density along the z' axis than within the average radius. Figure 16.3*a* illustrates that case. If $3z'^2 - r'^2$ is negative, the z' axis contains less of the nuclear charge density and there is a

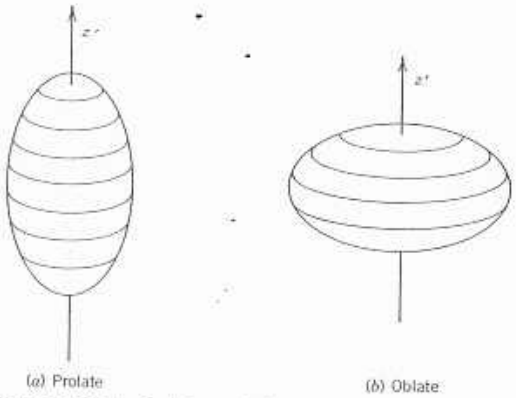

Figure 16.3 Prolate and oblate charge distributions.

corresponding flattening. In this case $Q < 0$ and the deformation is *oblate* (Figure 16.3*b*).

The energy of interaction of the nuclear charge distribution with an *externally supplied* (perhaps from the atomic electrons) potential V_{ext} is

$$E = \int \rho(\boldsymbol{r}') V_{ext}(\boldsymbol{r}')\, d\tau' \tag{16.23}$$

again integrated over the nuclear volume. (Consider how this reduces to the familiar expression for a point charge in an external field when V_{ext} = constant.) If we expand V_{ext} in a Taylor series about the center of the nucleus, then there is a constant term depending on $V_{ext}(0)$, which is of no interest, a dipole term which involves integrals such as

$$\int \rho(\boldsymbol{r}') z' \left(\frac{\partial V_{ext}}{\partial z'} \right)_{z'=0} d\tau'$$

which vanishes by the same parity argument presented above, and a nonvanishing quadrupole term, proportional to integrals of the form

$$\int \rho(\boldsymbol{r}') z'^2 \left(\frac{\partial^2 V_{ext}}{\partial z'^2} \right)_{z'=0} d\tau'$$

In all there are nine possible terms (involving x'^2, $x'y'$, etc.). If the external field has cylindrical symmetry (as in many cases of interest for atoms), then we can reduce the electric quadrupole contribution to the energy to the following form:

$$E_Q = \tfrac{1}{4}(eQ')\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)\left(\frac{\partial^2 V_{ext}}{\partial z^2} \right)_{z=0} \tag{16.24}$$

where θ is now the angle between the symmetry axis (now the z axis) of V_{ext} and the nuclear symmetry axis. The quadrupole moment Q' is calculated with respect to the z axis (the symmetry direction of V_{ext}), while Q of Equation 16.22 is calculated with respect to the nuclear symmetry direction z'. In evaluating Equation 16.24, we must take into account the directional relationships among

these different reference systems. The nuclear angular momentum has component I_z relative to the chosen z axis, and thus

$$\cos\theta = \frac{I_z}{|I|} = \frac{m_I}{\sqrt{I(I+1)}} \tag{16.25}$$

Evaluating the expression $eQ'(\frac{3}{2}\cos^2\theta - \frac{1}{2})$, with Q defined as always with respect to the axis of maximum projection of I_z, the result is

$$E_Q = \tfrac{1}{4}eQ\frac{3m_I^2 - I(I+1)}{I(2I-1)}\left(\frac{\partial^2 V_{\text{ext}}}{\partial z^2}\right)_{z=0} \tag{16.26}$$

In Section 16.3 we consider the case in which the angle θ is determined by the relationship between the nuclear spin I and atomic spin J.

16.3 HYPERFINE STRUCTURE

Hyperfine structure was originally taken to include those atomic effects (much smaller than the fine structure) that arise from the coupling between the electronic and nuclear angular momenta. It is thus an "internal" effect in atoms, and we cannot switch it off or modify it except by changing the nuclear or electronic structure (going to excited states, for instance). These effects were first studied by optical spectroscopists, who observed them as small perturbations in the structure of spectral lines. Modern techniques using lasers have extended these measurements to unprecedented levels of precision.

In recent years, hyperfine structure has come to include all effects that originate with the coupling of nuclear spins and moments with their environment, including the atomic electrons. The environment is often under the direct control of the experimenter, who can alter the hyperfine structure by, for example, changing an externally applied magnetic field. In this section we adopt this broad interpretation of hyperfine interactions.

Atomic states are labeled using the spectroscopic notation $n^{2S+1}L_J$, in which L is indicated by the usual designation S, P, D, F, ... corresponding to $L = 0, 1, 2, 3, \ldots$. For atomic states with only a single electron, such as the alkali atoms, the atomic spectroscopic notation is similar to the conventional notation used to designate individual electron states. Thus the sodium ground state, with its $3s_{1/2}$ electron, would be represented as $^2S_{1/2}$. The principal quantum number n is often not indicated.

We will use I to represent the total nuclear angular momentum (the nuclear spin). Similarly, J will represent the total (intrinsic plus orbital) electronic angular momentum. In ideal hydrogenic atoms, the electron moves in the nuclear Coulomb potential in quantum states of well defined orbital angular momentum L. Including the electron spin gives a second label S. In principle, it should not matter whether we label the electronic states of this ideal atom by the set of quantum numbers L, m_L, S, m_S or the set L, S, J, m_J. However, the *spin-orbit* interaction, which produces the *fine structure* of electronic levels, couples L and S in such a way that m_L and m_S are no longer well-defined, and the coupling

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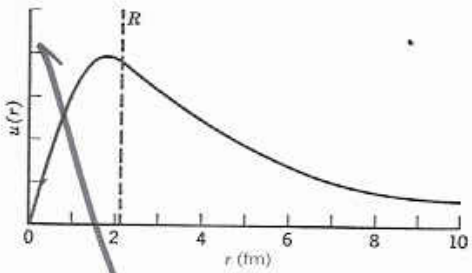


Figure 4.2 The deuteron wave function for $R = 2.1$ fm. Note how the exponential joins smoothly to the sine at $r = R$, so that both $u(r)$ and du/dr are continuous. If the wave function did not "turn over" inside $r = R$, it would not be possible to connect smoothly to a decaying exponential (negative slope) and there would be no bound state.

function finite for $r \to \infty$ we must have $D = 0$, and to keep it finite for $r \to 0$ we must have $B = 0$. (ψ depends on $u(r)/r$; as $r \to 0$, $u(r)$ also must go to zero.) Applying the continuity conditions on u and du/dr at $r = R$, we obtain

$$k_1 \cot k_1 R = -k_2 \tag{4.5}$$

This transcendental equation gives a relationship between V_0 and R. From electron scattering experiments, the rms charge radius of the deuteron is known to be about 2.1 fm, which provides a reasonable first estimate for R. Solving Equation 4.5 numerically (see Problem 6 at the end of this chapter) the result is $V_0 = 35$ MeV. This is actually quite a reasonable estimate of the strength of the nucleon–nucleon potential, even in more complex nuclei. (Note, however, that the proton and neutron are very likely to be found at separations greater than R; see Problem 4.)

We can see from Figure 4.1 how close the deuteron is to the top of the well. If the nucleon–nucleon force were just a bit weaker, the deuteron bound state would not exist (see Problem 3). We are fortunate that it does, however, because the formation of deuterium from hydrogen is the first step not only in the proton-proton cycle of fusion by which our sun makes its energy, but also in the formation of stable matter from the primordial hydrogen that filled the early universe. If no stable two-nucleon bound state existed, we would not be here to discuss it! (For more on the cosmological consequences of the formation of deuterium in the early universe, see Chapter 19.)

The deuteron wave function is shown in Figure 4.2. The weak binding means that $\psi(r)$ is just barely able to "turn over" in the well so as to connect at $r = R$ with the negative slope of the decaying exponential.

DEUTERÓN

Spin and Parity

The total angular momentum I of the deuteron should have three components: the individual spins s_n and s_p of the neutron and proton (each equal to $\frac{1}{2}$), and the orbital angular momentum ℓ of the nucleons as they move about their common center of mass:

$$I = s_n + s_p + \ell \tag{4.6}$$

When we solved the Schrödinger equation for the deuteron, we assumed $\ell = 0$ in analogy with the lowest bound state (the 1s state) in atomic hydrogen. The measured spin of the deuteron is $I = 1$ (how this is measured is discussed in Chapter 16). Since the neutron and proton spins can be either parallel (for a total of 1) or antiparallel (for a total of zero), there are four ways to couple s_n, s_p, and ℓ to get a total I of 1:

(a) s_n and s_p parallel with $\ell = 0$,
(b) s_n and s_p antiparallel with $\ell = 1$,
(c) s_n and s_p parallel with $\ell = 1$,
(d) s_n and s_p parallel with $\ell = 2$.

Another property of the deuteron that we can determine is its *parity* (even or odd), the behavior of its wave function when $r \rightarrow -r$ (see Section 2.6). By studying the reactions involving deuterons and the property of the photon emitted during the formation of deuterons, we know that its parity is even. In Section 2.6 we discussed that the parity associated with orbital motion is $(-1)^{\ell}$, even parity for $\ell = 0$ (s states) and $\ell = 2$ (d states) and odd parity for $\ell = 1$ (p states). The observed even parity allows us to eliminate the combinations of spins that include $\ell = 1$, leaving $\ell = 0$ and $\ell = 2$ as possibilities. The spin and parity of the deuteron are therefore consistent with $\ell = 0$ as we assumed, but of course we cannot yet exclude the possibility of $\ell = 2$.

Magnetic Dipole Moment

In Section 3.5 we discussed the spin and orbital contributions to the magnetic dipole moment. If the $\ell = 0$ assumption is correct, there should be no orbital contribution to the magnetic moment, and we can assume the total magnetic moment to be simply the combination of the neutron and proton magnetic moments:

$$\begin{aligned} \mu &= \mu_n + \mu_p \\ &= \frac{g_{sn}\mu_N}{\hbar} s_n + \frac{g_{sp}\mu_N}{\hbar} s_p \end{aligned} \tag{4.7}$$

where $g_{sn} = -3.826084$ and $g_{sp} = 5.585691$. As we did in Section 3.5, we take the observed magnetic moment to be the z component of μ when the spins have their maximum value ($+\frac{1}{2}\hbar$):

$$\begin{aligned} \mu &= \tfrac{1}{2}\mu_N(g_{sn} + g_{sp}) \\ &= 0.879804\ \mu_N \end{aligned} \tag{4.8}$$

The observed value is $0.8574376 \pm 0.0000004\ \mu_N$, in good but not quite exact agreement with the calculated value. The small discrepancy can be ascribed to any of a number of factors, such as contributions from the mesons exchanged between the neutron and proton; in the context of the present discussion, we can assume the discrepancy to arise from a small mixture of d state ($\ell = 2$) in the deuteron wave function:

$$\psi = a_s \psi(\ell = 0) + a_d \psi(\ell = 2) \tag{4.9}$$

Calculating the magnetic moment from this wave function gives

$$\mu = a_s^2 \mu(\ell = 0) + a_d^2 \mu(\ell = 2) \tag{4.10}$$

where $\mu(\ell = 0)$ is the value calculated in Equation 4.8 and $\mu(\ell = 2) = \frac{1}{4}(3 - g_{sp} - g_{sp})\mu_N$ is the value calculated for a d state. The observed value is consistent with $a_s^2 = 0.96$, $a_d^2 = 0.04$; that is, the deuteron is 96% $\ell = 0$ and only 4% $\ell = 2$. The assumption of the pure $\ell = 0$ state, which we made in calculating the well depth, is thus pretty good but not quite exact.

Electric Quadrupole Moment

The bare neutron and proton have no electric quadrupole moment, and so any measured nonzero value for the quadrupole moment must be due to the orbital motion. Thus the pure $\ell = 0$ wave function would have a vanishing quadrupole moment. The observed quadrupole moment is

$$Q = 0.00288 \pm 0.00002 \text{ b}$$

which, while small by comparison with many other nuclei, is certainly not zero.

The mixed wave function of Equation 4.9, when used as in Equation 3.36 to evaluate Q, gives two contributions, one proportional to a_d^2 and another proportional to the cross-term $a_s a_d$. Performing the calculation we obtain

$$Q = \frac{\sqrt{2}}{10} a_s a_d \langle r^2 \rangle_{sd} - \frac{1}{20} a_d^2 \langle r^2 \rangle_{dd} \tag{4.11}$$

where $\langle r^2 \rangle_{sd} = \int r^2 R_s(r) R_d(r) r^2 \, dr$ is the integral of r^2 over the radial wave functions; $\langle r^2 \rangle_{dd}$ is similarly defined. To calculate Q we must know the deuteron d-state wave function, which is not directly measurable. Using the realistic phenomenological potentials discussed later in this chapter gives reasonable values for Q with d-state admixtures of several percent, consistent with the value of 4% deduced from the magnetic moment.

This good agreement between the d-state admixtures deduced from μ and Q should be regarded as a happy accident and not taken too seriously. In the case of the magnetic dipole moment, there is no reason to expect that it is correct to use the free-nucleon magnetic moments in nuclei. (In fact, in the next chapter we see that there is strong evidence to the contrary.) Unfortunately, a nucleon in a deuteron lies somewhere between a free nucleon and a strongly bound nucleon in a nucleus, and we have no firm clues about what values to take for the magnetic moments. Spin-orbit interactions, relativistic effects, and meson exchanges may have greater effects on μ than the d-state admixture (but may cancel one another's effects). For the quadrupole moment, the poor knowledge of the d-state wave function makes the deduced d-state admixture uncertain. (It would probably be more valid to regard the calculation of Q, using a known d-state mixture, as a test of the d-state wave function.) Other experiments, particularly scattering experiments using deuterons as targets, also give d-state admixtures in the range of 4%. Thus our conclusions from the magnetic dipole and electric quadrupole moments may be valid after all!

It is important that we have an accurate knowledge of the d-state wave function because the mixing of ℓ values in the deuteron is the best evidence we have for the noncentral (tensor) character of the nuclear force.

incorrect, to explain the nonexistence of the di-proton as arising from Coulomb repulsion. No such temptation exists for the di-neutron, the nonexistence of which must arise from the spin dependence of the nuclear interaction. Reviewing the evidence, we first learned that the deuteron ground state is a spin triplet and that no bound spin singlet state exists. We then argued that, because identical fermions must have total antisymmetric wave functions and because the lowest state is expected to be a spatially symmetric $\ell = 0$ state, the di-proton and di-neutron systems must have antisymmetric, or singlet, spin states which are unbound.)

4.4 PROPERTIES OF THE NUCLEAR FORCE

Based on the low-energy properties described in the previous sections, we can learn many details about the nuclear force. When we include results from higher energy experiments, still more details emerge. In this section we summarize the main features of the internucleon force and in the next section we discuss a particular representation for the force that reproduces many of these details.

The Interaction between Two Nucleons Consists to Lowest Order of an Attractive Central Potential

In this chapter we have used for this potential a square-well form, which simplifies the calculations and reproduces the observed data fairly well. Other more realistic forms could just as well have been chosen, but the essential conclusions would not change (in fact, the effective range approximation is virtually independent of the shape assumed for the potential). The common characteristic of these potentials is that they depend only on the internucleon distance r. We therefore represent this central term as $V_c(r)$. The experimental program to study $V_c(r)$ would be to measure the energy dependence of nucleon–nucleon parameters such as scattering phase shifts, and then to try to choose the form for $V_c(r)$ that best reproduces those parameters.

The Nucleon–Nucleon Interaction is Strongly Spin Dependent

This observation follows from the failure to observe a singlet bound state of the deuteron and also from the measured differences between the singlet and triplet cross sections. What is the form of an additional term that must be added to the potential to account for this effect? Obviously the term must depend on the spins of the two nucleons, s_1 and s_2, but not all possible combinations of s_1 and s_2 are permitted. The nuclear force must satisfy certain symmetries, which restrict the possible forms that the potential could have. Examples of these symmetries are *parity* ($r \rightarrow -r$) and *time reversal* ($t \rightarrow -t$). Experiments indicate that, to a high degree of precision (one part in 10^7 for parity and one part in 10^3 for time reversal), the internucleon potential is invariant with respect to these operations. Under the parity operator, which involves spatial reflection, angular momentum vectors are unchanged. This statement may seem somewhat surprising, because upon inverting a coordinate system we would naturally expect all vectors defined

in that coordinate system to invert. However, angular momentum is not a true or polar vector; it is a pseudo- or axial vector that does not invert when $r \rightarrow -r$. This follows directly from the definition $r \times p$ or can be inferred from a diagram of a spinning object. Under the time-reversal operation, all motions (including linear and angular momentum) are reversed. Thus terms such as s_1 or s_2 or a linear combination $As_1 + Bs_2$ in the potential would violate time-reversal invariance and cannot be part of the nuclear potential; terms such as s_1^2, s_2^2, or $s_1 \cdot s_2$ are invariant with respect to time reversal and are therefore allowed. (All of these terms are also invariant with respect to parity.) The simplest term involving both nucleon spins is $s_1 \cdot s_2$. Let's consider the value of $s_1 \cdot s_2$ for singlet and triplet states. To do this we evaluate the total spin $S = s_1 + s_2$

$$\begin{aligned} S^2 = S \cdot S &= (s_1 + s_2) \cdot (s_1 + s_2) \\ &= s_1^2 + s_2^2 + 2s_1 \cdot s_2 \end{aligned}$$

Thus

$$s_1 \cdot s_2 = \tfrac{1}{2}\left(S^2 - s_1^2 - s_2^2\right) \tag{4.44}$$

To evaluate this expression, we must remember that in quantum mechanics all squared angular momenta evaluate as $s^2 = \hbar^2 s(s+1)$; see Section 2.5 and Equation 2.69.

$$\langle s_1 \cdot s_2 \rangle = \tfrac{1}{2}\left[S(S+1) - s_1(s_1+1) - s_2(s_2+1)\right]\hbar^2 \tag{4.45}$$

With nucleon spins s_1 and s_2 of $\frac{1}{2}$, the value of $s_1 \cdot s_2$ is, for triplet ($S = 1$) states:

$$\langle s_1 \cdot s_2 \rangle = \tfrac{1}{2}\left[1(1+1) - \tfrac{1}{2}\left(\tfrac{1}{2}+1\right) - \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\right]\hbar^2 = \tfrac{1}{4}\hbar^2 \tag{4.46}$$

and for singlet ($S = 0$) states:

$$\langle s_1 \cdot s_2 \rangle = \tfrac{1}{2}\left[0(0+1) - \tfrac{1}{2}\left(\tfrac{1}{2}+1\right) - \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\right]\hbar^2 = -\tfrac{3}{4}\hbar^2 \tag{4.47}$$

Thus a spin-dependent expression of the form $s_1 \cdot s_2 V_s(r)$ can be included in the potential and will have the effect of giving different calculated cross sections for singlet and triplet states. The magnitude of V_s can be adjusted to give the correct differences between the singlet and triplet cross sections and the radial dependence can be adjusted to give the proper dependence on energy.

We could also write the potential including V_c and V_s as

$$V(r) = -\left(\frac{s_1 \cdot s_2}{\hbar^2} - \tfrac{1}{4}\right)V_1(r) + \left(\frac{s_1 \cdot s_2}{\hbar^2} + \tfrac{3}{4}\right)V_3(r) \tag{4.48}$$

where $V_1(r)$ and $V_3(r)$ are potentials that separately give the proper singlet and triplet behaviors.

The Internucleon Potential Includes a Noncentral Term, Known as a Tensor Potential

Evidence for the tensor force comes primarily from the observed quadrupole moment of the ground state of the deuteron. An s-state ($\ell = 0$) wave function is spherically symmetric; the electric quadrupole moment vanishes. Wave functions with mixed ℓ states must result from noncentral potentials. This tensor force

must be of the form $V(r)$, instead of $V(r)$. For a single nucleon, the choice of a certain direction in space is obviously arbitrary; nucleons do not distinguish north from south or east from west. The only reference direction for a nucleon is its spin, and thus only terms of the form $s \cdot r$ or $s \times r$, which relate r to the direction of s, can contribute. To satisfy the requirements of parity invariance, there must be an even number of factors of r, and so for two nucleons the potential must depend on terms such as $(s_1 \cdot r)(s_2 \cdot r)$ or $(s_1 \times r) \cdot (s_2 \times r)$. Using vector identities we can show that the second form can be written in terms of the first and the additional term $s_1 \cdot s_2$, which we already included in $V(r)$. Thus without loss of generality we can choose the tensor contribution to the internucleon potential to be of the form $V_T(r)S_{12}$, where $V_T(r)$ gives the force the proper radial dependence and magnitude, and

$$S_{12} = 3(s_1 \cdot r)(s_2 \cdot r)/r^2 - s_1 \cdot s_2 \tag{4.49}$$

which gives the force its proper tensor character and also averages to zero over all angles.

The Nucleon–Nucleon Force Is Charge Symmetric

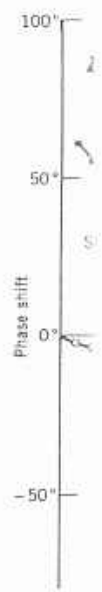
This means that the proton–proton interaction is identical to the neutron–neutron interaction, after we correct for the Coulomb force in the proton–proton system. Here "charge" refers to the character of the nucleon (proton or neutron) and not to electric charge. Evidence in support of this assertion comes from the equality of the pp and nn scattering lengths and effective ranges. Of course, the pp parameters must first be corrected for the Coulomb interaction. When this is done, the resulting singlet pp parameters are

$$a = -17.1 \pm 0.2 \text{ fm}$$
$$r_0 = 2.84 \pm 0.03 \text{ fm}$$

These are in very good agreement with the measured nn parameters ($a = -16.6 \pm 0.5$ fm, $r_0 = 2.66 \pm 0.15$ fm), which strongly supports the notion of charge symmetry.

The Nucleon–Nucleon Force Is Nearly Charge Independent

This means that (in analogous spin states) the three *nuclear* forces nn, pp, *and* pn are identical, again correcting for the pp Coulomb force. Charge independence is thus a stronger requirement than charge symmetry. Here the evidence is not so conclusive; in fact, the singlet np scattering length (-23.7 fm) seems to differ substantially from the pp and nn scattering lengths (-17 fm). However, we see from Figure 4.11 that large negative scattering lengths are extraordinarily sensitive to the nuclear wave function near $r = R$, and a very small change in ψ can give a large change in the scattering length. Thus the large difference between the scattering lengths may correspond to a very small difference (of order 1%) between the potentials, which (as we see in the next section) is easily explained by the exchange force model.



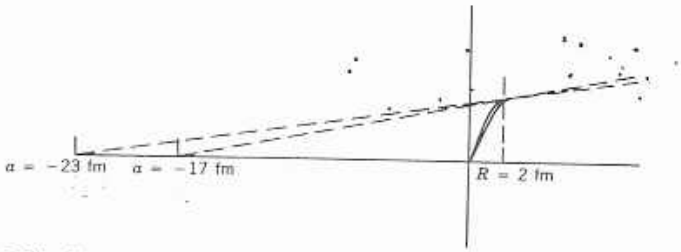


Figure 4.11 Very small changes in the nucleon–nucleon wave function near $r = R$ can lead to substantial differences in the scattering length when the extrapolation is made (compare Figure 4.7*b*).

The Nucleon–Nucleon Interaction Becomes Repulsive at Short Distances

This conclusion follows from qualitative considerations of the nuclear density: as we add more nucleons, the nucleus grows in such a way that its central density remains roughly constant, and thus something is keeping the nucleons from crowding too closely together. More quantitatively, we can study nucleon–nucleon scattering at higher energies. Figure 4.12 shows the deduced singlet s-wave phase shifts for nucleon–nucleon scattering up to 500 MeV. (At these energies, phase shifts from higher partial waves, p and d for example, also contribute to the cross

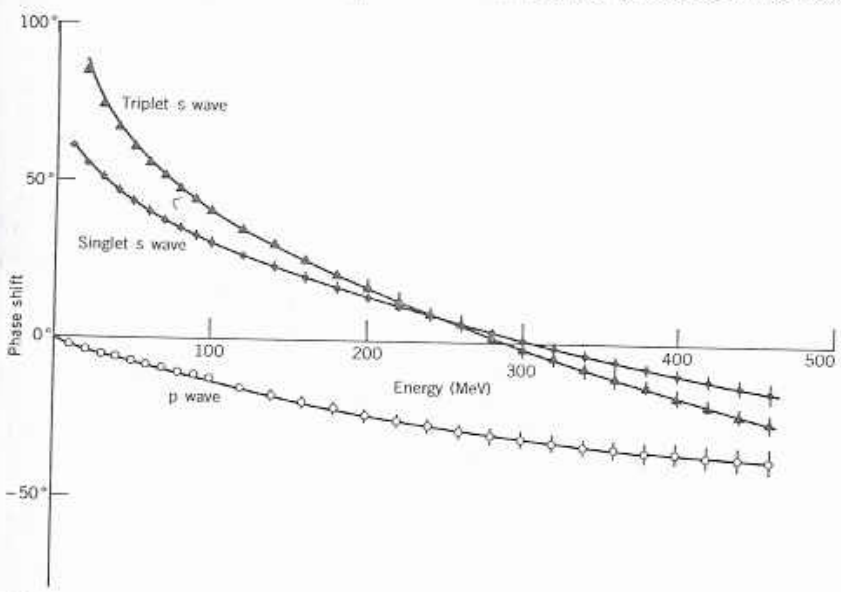


Figure 4.12 The phase shifts from neutron–proton scattering at medium energies. The change in the s-wave phase shift from positive to negative at about 300 MeV shows that at these energies the incident nucleon is probing a repulsive core in the nucleon–nucleon interaction. ▲, $^3S_1$; ●, $^1S_0$; O, $^1P_1$. Data from M. MacGregor et al., *Phys. Rev.* **182**, 1714 (1969).

$M = 2m_R$ is the nucleon mass. Neglecting magnetic moment terms, j is the convective current of the proton

$$j \simeq \frac{p}{2m_R} = \frac{p}{M} = \frac{v}{2} = v_p \tag{36}$$

v_p is the proton velocity. Averaging over photon polarizations, the photodisintegration cross section of the deuteron is found to be

$$\sigma = \frac{\alpha p}{4\pi\omega}\,\frac{4\pi}{3}\,|p_{fi}|^2 \tag{37}$$

We must now calculate the matrix element of the electric dipole; it equals

$$p_{fi} = \int d^3x\,\psi^*_f\, p\,\psi_i = \int d^3x\, e^{-ipx}\, p \sqrt{\frac{k}{2\pi}}\,\frac{e^{-kr}}{r} \tag{38}$$

Here, the expression of the deuteron wave function for $r > a$ was taken to be approximately valid everywhere, with due normalization. An integration by parts yields

$$p_{fi} = \sqrt{\frac{k}{2\pi}}\,\frac{4\pi p}{p^2 + k^2} \tag{39}$$

We make use of

$$E_B = \frac{k^2}{2\,m_R} \tag{40}$$

and neglect the nucleus recoil energy, $\omega^2/4m_R$, for $\omega \ll 4m_R$ with the result

$$\sigma = \frac{8\pi}{3}\,\frac{\alpha}{M}\,\sqrt{E_B}\,\frac{(\omega - E_B)^{\frac{3}{2}}}{\omega^3} \tag{41}$$

The cross section has its maximum at $\omega \simeq 2E_B$.

Problem 108: Some properties of the deuteron

The deuteron has isospin 0, angular momentum 1 and well-defined parity.

a) Which states are compatible with the above properties?

b) Knowing the following values of magnetic moment

$$\begin{aligned} \mu_p &= 2.7928\,\frac{e\hbar}{2m_pc} && \text{(proton)} \\ \mu_n &= -1.913\,\frac{e\hbar}{2m_pc} && \text{(neutron)} \\ \mu_D &= 0.8565\,\frac{e\hbar}{2m_pc} && \text{(deuteron)} \end{aligned} \tag{1}$$

deduce the state of the deuteron.

c) Consider elastic scattering of electrons off deuterons polarized transversely to the relative momentum. What is the effect of the magnetic dipole and of the quadrupole moment on the angular distribution?

Solution

a) By the Pauli exclusion principle, the wave function of two-nucleon states is completely antisymmetric. Consequently, the spin S, orbital angular momentum L and isospin T must satisfy: $S + T + L =$ odd.

Isospin singlet states of the n-p system with total angular momentum 1 will thus have the following quantum numbers

$$^3S_1^+ \qquad ^1P_1^- \qquad ^3D_1^+ \tag{2}$$

Here we have used the notation $^{2S+1}L_J^P$. We see that the deuteron can be either a mixture of $^3S_1$ and $^3D_1$ or a pure $^1P_1$ state.

b) The magnetic moment operator for a system of two spin-1/2 particles, expressed in units of the nuclear magneton $e\hbar/2m_pc$, equals

$$\mathbf{M} = \mu_p\,\boldsymbol{\sigma}_p + \mu_n\,\boldsymbol{\sigma}_n + \frac{\mathbf{L}}{2} \tag{3}$$

The factor of 1/2 in the orbital term is due to the fact that only one of the two particles (the proton) is charged. The magnetic moment must be parallel to the angular momentum $\mathbf{J} = \mathbf{L} + \boldsymbol{\sigma}_p/2 + \boldsymbol{\sigma}_n/2$, by the Wigner- Eckart theorem. We define μ_D through the relationship

$$\mathbf{M} = \mu_D \mathbf{J} \tag{4}$$

equivalently

$$\mu_D = \frac{\mathbf{M}\cdot\mathbf{J}}{\mathbf{J}\cdot\mathbf{J}} \tag{5}$$

Writing $\mathbf{M}$ in the form

$$\mathbf{M} = (\mu_p + \mu_n)\,\mathbf{S} + \frac{\mathbf{L}}{2} + (\mu_p - \mu_n)\,(\mathbf{S}_p - \mathbf{S}_n) \tag{6}$$

and substituting into (5) we find

$$\mu_D = \frac{1}{2}\left[\mu_p + \mu_n + \frac{1}{2} + \left(\mu_p + \mu_n - \frac{1}{2}\right)\frac{\mathbf{S}^2 - \mathbf{L}^2}{2}\right] \tag{7}$$

The following scalar products have been used in deriving (7)

$$\mathbf{S}\cdot\mathbf{J} = \frac{\mathbf{J}^2 + \mathbf{S}^2 - \mathbf{L}^2}{2} \quad , \quad \mathbf{L}\cdot\mathbf{J} = \frac{\mathbf{J}^2 + \mathbf{L}^2 - \mathbf{S}^2}{2} \tag{8}$$

and

$$(\mathbf{S}_p - \mathbf{S}_n)\cdot\mathbf{J} = (\mathbf{S}_p - \mathbf{S}_n)\cdot\mathbf{L} = 0 \tag{9}$$

Eq.(9) is a consequence of the fact that the only nonzero matrix elements of the operator $\mathbf{S}_p - \mathbf{S}_n$ occur in the singlet-triplet transition.

The numerical values of μ_D can now be read off (7)

$$\mu_D\left({}^3S_1\right) = 0.8798 \tag{10}$$
$$\mu_D\left({}^3D_1\right) = 0.3101 \tag{11}$$
$$\mu_D\left({}^1P_1\right) = 0.5 \tag{12}$$

A comparison of the above with the experimental result excludes the state ${}^1P_1$, so that the deuteron must be a superposition of the following form

$$|d\rangle = \cos\alpha\,|{}^3S_1\rangle + \sin\alpha\,|{}^3D_1\rangle \tag{13}$$

The magnetic moment of this state is

$$\mu_D = \cos^2\alpha\,\mu_D({}^3S_1) + \sin^2\alpha\,\mu_D({}^3D_1) \tag{14}$$

The experimental value of μ_D now fixes α

$$\cos\alpha = 0.98 \qquad \sin\alpha = 0.2 \tag{15}$$

c) The expression for the electron-deuteron scattering amplitude is

$$\mathcal{M}_{fi} = \bar{u}(\mathrm{p}')\gamma_\mu u(\mathrm{p})\frac{e^2}{q^2}J_d^\mu(\mathrm{q}) \tag{16}$$

$J_d^\mu(\mathrm{q}) = \langle d_f|j^\mu(0)|d_i\rangle$ is the deuteron electromagnetic current.

In the nonrelativistic limit, the contribution of this current to the amplitude becomes

$$J_d^0(\mathrm{q}) = 2m_D\left[\varepsilon'^*\varepsilon + \frac{Q}{2}\left\{(\varepsilon'^*\,\mathrm{q})\,(\varepsilon\,\mathrm{q}) - \frac{\mathrm{q}^2}{2}\,(\varepsilon'^*\,\varepsilon)\right\}\right] \tag{17}$$
$$\mathbf{J}_d = 2m_D\mathrm{q}\wedge\mu \tag{18}$$

where $Q = Q_{zz}/2$ is the quadrupole moment of the deuteron and $\mu = \mu\,(\varepsilon'^*\wedge\varepsilon)$ is the magnetic moment; μ is given in units of $e\hbar/2m_Dc$.

To derive the angular distribution we evaluate

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{(q^2)^2}\,L_{\mu\nu}\,\overline{J_d^{\mu*}J_d^\nu} \tag{19}$$

where

$$L_{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}\left[\left(\not{p}+m\right)\gamma_\mu\left(\not{p}'+m\right)\gamma_\nu\right] = q^2 g_{\mu\nu} - P_\mu P_\nu \tag{20}$$
$$\overline{J_d^{\mu*}J_d^\nu} = \sum_{\text{final pol.}} J_d^{\mu*}J_d^\nu$$
$$P_\mu = p_\mu + p'_\mu \qquad q_\mu = p_\mu - p'_\mu$$

In the limit of small momentum transfer we have

$$L_{00} = 4p_0^2 - \mathrm{q}^2 \qquad L_{ij} = \mathrm{q}^2\delta_{ij} + 4p_ip_j \qquad L_{0i} = 4p_0p_i \tag{21}$$

We perform the sum over final polarizations, λ, using

$$\sum_\lambda \varepsilon'_i(\lambda)\varepsilon'^*_j(\lambda) = \delta_{ij} \tag{22}$$

with the result

$$\overline{J_d^{0*}J_d^0} = (2m_D)^2\left\{\left(1-\frac{Q}{6}\mathrm{q}^2\right)^2 + |\varepsilon\,\mathrm{q}|^2Q^2\left(1+\frac{1}{12}Q^2\mathrm{q}^2\right)\right\}$$
$$\overline{J_d^{0*}\mathbf{J}^*} = (2m_D)^2\left\{\left[\varepsilon\,(\mathrm{q}\,\varepsilon^*) - \varepsilon^*\,(\mathrm{q}\,\varepsilon)\right]\left(1-Q\frac{\mathrm{q}^2}{6}\right) + \frac{1}{Q}\,Q\left(|\mathrm{q}\,\varepsilon|^2\mathrm{q} - (\mathrm{q}\,\varepsilon)\varepsilon^*\,\mathrm{q}^2\right)\right\} \tag{23}$$
$$\overline{J_i^*J_j} = (2m_D)^2\left\{\delta_{ij}\,|\mathrm{q}\,\varepsilon|^2 + \varepsilon_i^*\varepsilon_j\,\mathrm{q}^2 - \varepsilon_i^*q_j(\mathrm{q}\,\varepsilon) - \varepsilon_jq_i(\mathrm{q}\,\varepsilon)^*\right\}$$

Using (21) and (23), the squared amplitude reads

$$\begin{aligned}|\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4}(2m_D)^2\Big[&\left(4p_0^2-\mathrm{q}^2\right)\left\{\left(1-\frac{Q}{6}\mathrm{q}^2\right)^2 + |\varepsilon\,\mathrm{q}|^2Q^2\left(1+\frac{1}{12}Q^2\mathrm{q}^2\right)\right\}\\ &+2\mu\,\mathrm{Re}\Big\{2p_0\left[(\mathrm{p}\,\varepsilon)\,(\mathrm{q}\,\varepsilon^*) - (\mathrm{p}\,\varepsilon^*)\,(\mathrm{q}\,\varepsilon)\right]\left(1-Q\frac{\mathrm{q}^2}{6}\right)\\ &+2p_0Q\left[|\mathrm{q}\,\varepsilon|^2\,(\mathrm{q}\,\mathrm{p}) - \mathrm{q}^2(\mathrm{q}\,\varepsilon)(\varepsilon^*\,\mathrm{p})\right]\Big\}\\ &+\mu^2\Big\{\mathrm{q}^2\left(|\varepsilon\,\mathrm{p}|^2+\mathrm{q}^2\right) + 4(\mathrm{p}^2|\varepsilon\,\mathrm{q}|^2 + \mathrm{q}^2|\varepsilon\,\mathrm{p}|^2\\ &-(\varepsilon\,\mathrm{p})(\varepsilon^*\,\mathrm{q})(\mathrm{p}\,\mathrm{q}) - (\varepsilon\,\mathrm{q})(\varepsilon^*\,\mathrm{p})(\mathrm{p}\,\mathrm{q})\Big)\Big\}\Big]\end{aligned} \tag{24}$$

If the polarization is perpendicular to the incident momentum, $\varepsilon\,\mathrm{p} = 0$, the above expression simplifies to

$$\begin{aligned}|\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4}(2m_D)^2\Big[&\left(4p_0^2-\mathrm{q}^2\right)\left\{\left(1-\frac{Q}{6}\mathrm{q}^2\right)^2 + |\varepsilon\,\mathrm{q}|^2Q^2\left(1+\frac{1}{12}Q^2\mathrm{q}^2\right)\right\}\\ &+\mu^2\mathrm{q}^2\left(\mathrm{q}^2+|\varepsilon\,\mathrm{q}|^2\right)\Big]\end{aligned} \tag{25}$$

The magnetic dipole and the quadrupole moment introduce a dependence on the angle between momentum transfer and polarization. Setting the polarization along the x-axis, we write

$$\begin{aligned}\mathrm{p} &= (0,0,p)\\ \varepsilon &= (1,0,0)\\ \mathrm{q} &= \mathrm{p}'-\mathrm{p} = p\,(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta-1)\\ \varepsilon\,\hat{\mathrm{q}} &= \cos\theta\cos\phi\end{aligned} \tag{26}$$

(44)

Modelo de capas

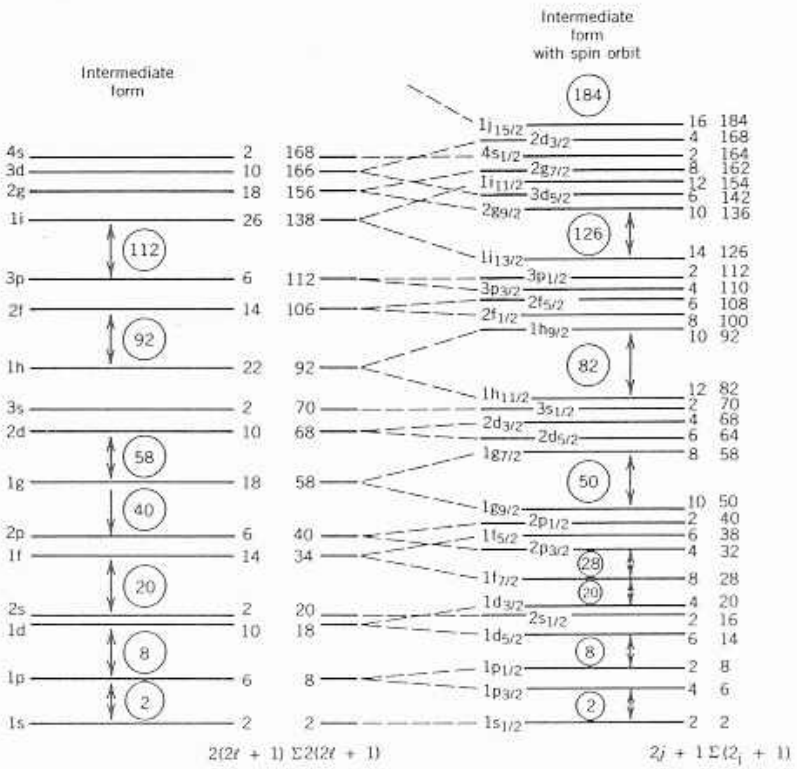


Figure 5.6 At the left are the energy levels calculated with the potential of Figure 5.5. To the right of each level are shown its capacity and the cumulative number of nucleons up to that level. The right side of the figure shows the effect of the spin-orbit interaction, which splits the levels with $\ell > 0$ into two new levels. The shell effect is quite apparent, and the magic numbers are exactly reproduced.

Spin-Orbit Potential (leer)

"ajuste de capas"

How can we modify the potential to give the proper magic numbers? We certainly cannot make a radical change in the potential, because we do not want to destroy the physical content of the model—Equation 5.1 is already a very good guess at how the nuclear potential *should* look. It is therefore necessary to add various terms to Equation 5.1 to try to improve the situation. In the 1940s, many unsuccessful attempts were made at finding the needed correction; success was finally achieved by Mayer, Haxel, Suess, and Jensen who showed in 1949 that the inclusion of a *spin-orbit* potential could give the proper separation of the subshells.

Once again, we are borrowing an idea from our colleagues, the atomic physicists. In atomic physics the spin-orbit interaction, which causes the observed fine structure of spectral lines, comes about because of the electromagnetic interaction of the electron's magnetic moment with the magnetic field generated by its motion about the nucleus. The effects are typically very small, perhaps one

part in 10^5 in the spacing of atomic levels. No such electromagnetic interaction would be strong enough to give the substantial changes in the *nuclear* level spacing needed to generate the observed magic numbers. Nevertheless we adopt the concept of a *nuclear* spin-orbit force of the same form as the atomic spin-orbit force but certainly *not* electromagnetic in origin. In fact, we know from the scattering experiments discussed in Chapter 4 that there is strong evidence for a nucleon–nucleon spin-orbit force.

The spin-orbit interaction is written as $V_{so}(r)\boldsymbol{\ell}\cdot\boldsymbol{s}$, but the form of $V_{so}(r)$ is not particularly important. It is the $\boldsymbol{\ell}\cdot\boldsymbol{s}$ factor that causes the reordering of the levels. As in atomic physics, in the presence of a spin-orbit interaction it is appropriate to label the states with the *total angular momentum* $\boldsymbol{j} = \boldsymbol{\ell} + \boldsymbol{s}$. A single nucleon has $s = \frac{1}{2}$, so the possible values of the total angular momentum quantum number are $j = \ell + \frac{1}{2}$ or $j = \ell - \frac{1}{2}$ (except for $\ell = 0$, in which case only $j = \frac{1}{2}$ is allowed). The expectation value of $\boldsymbol{\ell}\cdot\boldsymbol{s}$ can be calculated using a common trick. We first evaluate $\boldsymbol{j}^2 = (\boldsymbol{\ell} + \boldsymbol{s})^2$:

$$\begin{aligned} \boldsymbol{j}^2 &= \boldsymbol{\ell}^2 + 2\boldsymbol{\ell}\cdot\boldsymbol{s} + \boldsymbol{s}^2 \\ \boldsymbol{\ell}\cdot\boldsymbol{s} &= \tfrac{1}{2}\left(\boldsymbol{j}^2 - \boldsymbol{\ell}^2 - \boldsymbol{s}^2\right) \end{aligned} \tag{5.2}$$

Putting in the expectation values gives

$$\langle \boldsymbol{\ell}\cdot\boldsymbol{s} \rangle = \tfrac{1}{2}\left[j(j+1) - \ell(\ell+1) - s(s+1) \right]\hbar^2 \tag{5.3}$$

Consider a level such as the 1f level ($\ell = 3$), which has a degeneracy of $2(2\ell + 1) = 14$. The possible j values are $\ell \pm \frac{1}{2} = \frac{5}{2}$ or $\frac{7}{2}$. Thus we have the levels $1f_{5/2}$ and $1f_{7/2}$. The degeneracy of each level is $(2j + 1)$, which comes from the m_j values. (With spin-orbit interactions, m_s and m_ℓ are no longer "good" quantum numbers and can no longer be used to label states or to count degeneracies.) The capacity of the $1f_{5/2}$ level is therefore 6 and that of $1f_{7/2}$ is 8, giving again 14 states (the number of possible states must be preserved; we are only grouping them differently). For the $1f_{5/2}$ and $1f_{7/2}$ states, which are known as a spin-orbit pair or doublet, there is an energy separation that is proportional to the value of $\langle \boldsymbol{\ell}\cdot\boldsymbol{s} \rangle$ for each state. Indeed, for any pair of states with $\ell > 0$, we can compute the energy difference using Equation 5.3:

$$\langle \boldsymbol{\ell}\cdot\boldsymbol{s} \rangle_{j=\ell+1/2} - \langle \boldsymbol{\ell}\cdot\boldsymbol{s} \rangle_{j=\ell-1/2} = \tfrac{1}{2}(2\ell + 1)\hbar^2 \tag{5.4}$$

The energy splitting increases with increasing ℓ. Consider the effect of choosing $V_{so}(r)$ to be negative, so that the member of the pair with the larger j is pushed downward. Figure 5.6 shows the effect of this splitting. The $1f_{7/2}$ level now appears in the gap between the second and third shells; its capacity of 8 nucleons gives the magic number 28. (The p and d splittings do not result in any major regrouping of the levels.) The next major effect of the spin-orbit term is on the 1g level. The $1g_{9/2}$ state is pushed down all the way to the next lower major shell; its capacity of 10 nucleons adds to the previous total of 40 for that shell to give the magic number of 50. A similar effect occurs at the top of each major shell. In each case the lower energy member of the spin-orbit pair from the next shell is pushed down into the lower shell, and the remaining magic numbers follow exactly as expected. (We even predict a new one, at 184, which has not yet been seen.)

As an example of the application of the shell model, consider the filling of levels needed to produce $^{15}_{8}\mathrm{O}$ and $^{17}_{8}\mathrm{O}$. The 8 protons fill a major shell and do not

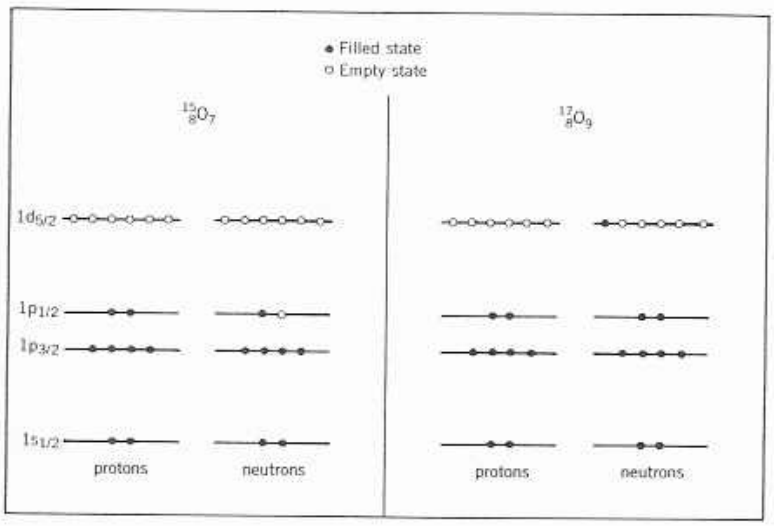


Figure 5.7 The filling of shells in ^{15}O and ^{17}O. The filled proton shells do not contribute to the structure; the properties of the ground state are determined primarily by the odd neutron.

contribute to the structure. Figure 5.7 shows the filling of levels. The extreme limit of the shell model asserts that only the single unpaired nucleon determines the properties of the nucleus. In the case of ^{15}O, the unpaired neutron is in the $p_{1/2}$ shell; we would therefore predict that the ground state of ^{15}O has spin $\frac{1}{2}$ and odd parity, since the parity is determined by $(-1)^{\ell}$. The ground state of ^{17}O should be characteristic of a $d_{5/2}$ neutron with spin $\frac{5}{2}$ and even parity. These two predictions are in exact agreement with the observed spin-parity assignments, and in fact similar agreements are found throughout the range of odd-A nuclei where the shell model is valid (generally $A < 150$ and $190 < A < 220$, for reasons to be discussed later in this chapter). This success in accounting for the observed ground-state spin-parity assignments was a great triumph for the shell model.

Magnetic Dipole Moments (IMPORTANTE) !!

Another case in which the shell model gives a reasonable (but not so exact) agreement with observed nuclear properties is in the case of magnetic dipole moments. You will recall from Chapter 3 that the magnetic moment is computed from the expectation value of the magnetic moment operator in the state with maximum z projection of angular momentum. Thus, including both ℓ and s terms, we must evaluate

$$\mu = \mu_N(g_\ell \ell_z + g_s s_z)/h \tag{5.5}$$

when $j_z = jh$. This cannot be evaluated directly, since ℓ_z and s_z do not have precisely defined values when we work in a system in which j is precisely defined. We can rewrite this expression, using $\mathbf{j} = \boldsymbol{\ell} + \mathbf{s}$, as

$$\mu = \left[g_\ell j_z + (g_s - g_\ell)\vec{s}_z\right]\mu_N/h \tag{5.6}$$

and, taking the expectation value when $j_z = jh$, the result is

$$\langle\mu\rangle = \left[g_\ell j + (g_s - g_\ell)\langle s_z\rangle/h\right]\mu_N \tag{5.7}$$

→ momentos magnéticos en el modelo de capas

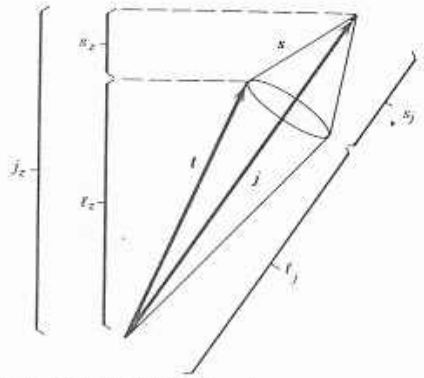


Figure 5.8 As the total angular momentum $\boldsymbol{j}$ precesses about the z axis keeping j_z constant, the vectors $\boldsymbol{\ell}$ and $\boldsymbol{s}$ precess about $\boldsymbol{j}$. The components of $\boldsymbol{\ell}$ and $\boldsymbol{s}$ along $\boldsymbol{j}$ remain constant, but ℓ_z and s_z vary.

The expection value of $\langle s_z \rangle$ can be quickly computed by recalling that $\boldsymbol{j}$ is the only vector of interest in this problem—the $\boldsymbol{\ell}$ and $\boldsymbol{s}$ vectors are meaningful only in their relationship to $\boldsymbol{j}$. Specifically, when we compute $\langle s_z \rangle$ the only surviving part will be from the component of $\boldsymbol{s}$ along $\boldsymbol{j}$, as suggested by the vector diagram of Figure 5.8. The instantaneous value of s_z varies, but its component along $\boldsymbol{j}$ remains constant. We therefore need an expression for the vector $\boldsymbol{s}_j$, the component of $\boldsymbol{s}$ along $\boldsymbol{j}$. The unit vector along $\boldsymbol{j}$ is $\boldsymbol{j}/|\boldsymbol{j}|$, and the component of $\boldsymbol{s}$ along $\boldsymbol{j}$ is $|\boldsymbol{s}\cdot\boldsymbol{j}|/|\boldsymbol{j}|$. The vector $\boldsymbol{s}_j$ is therefore $\boldsymbol{j}|\boldsymbol{s}\cdot\boldsymbol{j}|/|\boldsymbol{j}|^2$, and replacing all quantities by their expectation values gives

$$\langle s_z \rangle = \frac{j}{2j(j+1)}\left[j(j+1) - \ell(\ell+1) + s(s+1)\right]\hbar \qquad (5.8)$$

where $\boldsymbol{s}\cdot\boldsymbol{j} = \boldsymbol{s}\cdot(\boldsymbol{\ell}+\boldsymbol{s})$ is computed using Equation 5.3. Thus for $j = \ell + \frac{1}{2}$, $\langle s_z \rangle = \hbar/2$, while for $j = \ell - \frac{1}{2}$ we have $\langle s_z \rangle = -\hbar j/2(j+1)$. The corresponding magnetic moments are

$$\begin{aligned} j &= \ell + \tfrac{1}{2} \qquad \langle \mu \rangle = \left[g_\ell\left(j - \tfrac{1}{2}\right) + \tfrac{1}{2}g_s\right]\mu_N \\ j &= \ell - \tfrac{1}{2} \qquad \langle \mu \rangle = \left[g_\ell\frac{j(j+\frac{3}{2})}{(j+1)} - \frac{1}{2}\frac{1}{j+1}g_s\right]\mu_N \end{aligned} \qquad (5.9)$$

Figure 5.9 shows a comparison of these calculated values with measured values for shell-model odd-A nuclei. The computed values are shown as solid lines and are known as the Schmidt lines; this calculation was first done by Schmidt in 1937. The experimental values fall within the limits of the Schmidt lines, but are generally smaller in magnitude and have considerable scatter. One defect of this theory is the assumption that g_s for a nucleon in a nucleus is the same as g_s for a free nucleon. We discussed in Chapter 3 how the spin g factors of nucleons differ considerably from the value of 2 expected for "elementary" spin-$\frac{1}{2}$ particles. If we regard the substantial differences as arising from the "meson cloud" that surrounds the nucleon, then it is not at all surprising that the meson cloud in nuclei, where there are other surrounding nucleons and mesons, differs from what it is for free nucleons. It is customary to account for this effect by (somewhat

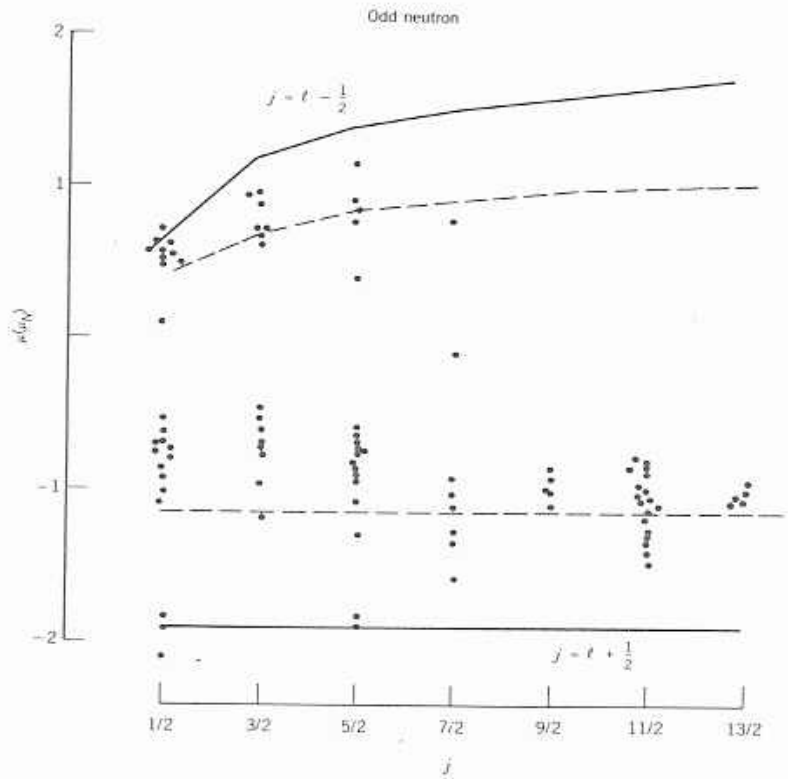


Figure 5.9 Experimental values for the magnetic moments of odd-neutron and odd-proton shell-model nuclei. The Schmidt lines are shown as solid for $g_s = g_s$(free) and dashed for $g_s = 0.6g_s$(free).

arbitrarily) reducing the g_s factor; for example, the lines for $g_s = 0.6g_s$(free) are shown in Figure 5.9. The overall agreement with experiment is better, but the scatter of the points suggests that the model is oversimplifying the calculation of magnetic moments. Nevertheless, the success in indicating the general trend of the observed magnetic moments suggests that the shell model gives us at least an approximate understanding of the structure of these nuclei.

Electric Quadrupole Moments

The calculation of electric quadrupole moments in the shell model is done by evaluating the electric quadrupole operator, $3z^2 - r^2$, in a state in which the total angular momentum of the odd particle has its maximum projection along the z axis (that is, $m_j = +j$). Let's assume for now that the odd particle is a proton. If its angular momentum is aligned (as closely as quantum mechanics allows) with the z axis, then it must be orbiting mostly in the xy plane. As we indicated in the discussion following Equation 3.36, this would give a negative quadrupole