tum $-\mathbf{p}$, energy E, but then applying parity and reversing all spatial directions puts it back to momentum $\mathbf{p}$, energy E. PT does affect the polarization of the photon though, and also the spin states. Note that at one end of the process we must make the inverse transformation, i.e. a $(PT)^{-1}$ transformation. Although this sounds the same as PT, there is a subtle difference as we shall see in a minute. Hence the C that changes from particle to antiparticle is equivalent to a parity reversal P together with a time reversal T. Everything is done in the reverse order in time–for example, if you have circularly polarized light, the polarization vector is say $(e_x, e_y) = (1, \mathrm{i})$, the time reversed polarization is $(e_x, e_y)^* = (1, -\mathrm{i})$ which has the electric vector going round in the reverse direction. Then $PT(e_x, e_y) = -(e_x, e_y)^*$ and so on. $C = PT$–everything backwards in time and reversed in space. I'm not going to go through the details to prove it though.

As mentioned above, when getting the particle behavior from the antiparticle, one spin state at one end has PT applied, the other at the other end has $(PT)^{-1}$ applied. We would prefer to have the same transformation applied to both, because if the spin states u_{i} and u_{f} are the same, then so are the spin states $(PT)u_{\mathrm{i}}$ and $(PT)u_{\mathrm{f}}$. We will need

to use this later. It turns out that there is no problem with the parity operation P, so let us choose the phases so that $P^2 = 1$, i.e. two space inversions is the same as doing nothing. What we are going to show though is that for spin $\frac{1}{2}$ particles $T^{-1} = -T$, i.e. that $TT = -1$, whereas for spin-zero $TT = +1$. That difference in sign, that extra minus sign, is where the Pauli exclusion principle and Fermi statistics come from.

THE EFFECT OF TWO SUCCESSIVE TIME REVERSALS

Why should it be that two time reversals change the sign of a spin $\frac{1}{2}$ particle? The answer is that changing T twice is equivalent to a 360° rotation. If I flipped the x-axis twice, I would be rotating through 360°, and thinking in four-dimensional spacetime; the same could be true of the t-axis too. Indeed it is true as I will demonstrate below (even without implying any relativistic relation of t and x!). Then, as we said above, rotating a spin $\frac{1}{2}$ particle by 360° multiplies it by (-1), so we find $TT = -1$. Let's show that we must have $TT = -1$ for spin $\frac{1}{2}$.

In Table 1 are listed various states, together with what you get if you apply T once, and then once

Table 1. *The effect of time reversal on various states.*

State: $\|a\rangle$	Time reversed: $T\|a\rangle$	Twice time reversed: $TT\|a\rangle$
$\|x\rangle$	$\|x\rangle$	$\|x\rangle$
$\|p\rangle = \Sigma e^{ipx}\|x\rangle$	$\|-p\rangle = \Sigma e^{-ipx}\|x\rangle$	$\|p\rangle$
$\alpha\|a\rangle + \beta\|b\rangle$	$\alpha^* T\|a\rangle + \beta^* T\|b\rangle$	$\alpha TT\|a\rangle + \beta TT\|b\rangle$
Integral spin states		
$\|j, m = 0\rangle$	$e^{i\phi}\|j, m = 0\rangle$	$e^{i\phi}(e^{-i\phi}\|j, m = 0\rangle) = \|j, m = 0\rangle$
Spin $\frac{1}{2}$ states		
$\|+z\rangle$	$\|-z\rangle$	$-\|+z\rangle$
$\|-z\rangle$	$-\|+z\rangle$	$-\|-z\rangle$
$\|+x\rangle = (\|+z\rangle + \|-z\rangle)/\sqrt{2}$	$(\|-z\rangle - \|+z\rangle)/\sqrt{2} = -\|-x\rangle$	$-\|+x\rangle$
$\|-x\rangle = (\|+z\rangle - \|-z\rangle)/\sqrt{2}$	$(\|-z\rangle + \|+z\rangle)/\sqrt{2} = \|+x\rangle$	$-\|-x\rangle$

more. The first state is the state where a particle is at the point x in space; this state is written $|x\rangle$ using Dirac's notation. In between the '|' and the '$\rangle$' one puts the name of the state, or just something to label it, which in this case is the point x where the state is. Then the time reversed state is $T|x\rangle = |x\rangle$, i.e. the particle will be at the same point, no big deal. On the other hand, a particle in a state of momentum p (i.e. in a state $|p\rangle$) will time reverse into a state of momentum $-p$, but then back to $|p\rangle$ with the second time reversal.

Considering the state $|p\rangle$ shows us that T is what is called an 'antiunitary' operation. $|p\rangle$ can be made by combining states $|x\rangle$ at different positions with different phases. To get the time reversed state $|-p\rangle$ just take the states $T|x\rangle = |x\rangle$ but with the complex conjugate of the phases used to construct $|p\rangle$. So in general $T[\alpha|a\rangle + \beta|b\rangle] = \alpha^* T|a\rangle + \beta^* T|b\rangle$, i.e. for an antiunitary operation you must take the complex conjugates of the coefficients whenever you see them. Of course, if you apply T again, you take the complex conjugate of the coefficients again, and if you are very good at algebra you know that doing that is a waste of time. Now $TT|a\rangle$ must be the same physical state $|a\rangle$, but the damn quantum mechanics always allows you to have a different phase. So, by the above argument, $TT|a\rangle$ = phase $|a\rangle$ with the same phase for all states that could be superposed with the state $|a\rangle$, so that any interference between states is the same before and after applying TT. Spin-zero and spin $\frac{1}{2}$ states cannot be superposed, the two sorts of state are fundamentally different; hence the overall phase change when you apply TT can be different between the two.

What we are going to use now is that if you have a state of angular momentum $|j, m\rangle$, then $T|j, m\rangle$ = phase $|j, -m\rangle$. It must be like this for angular momentum: the time reverse of something spinning one way is the object spinning in the opposite direction. For example, with orbital angular momentum $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$, we find that since T sends $\mathbf{r} \to \mathbf{r}$ and $\mathbf{p} \to -\mathbf{p}$, then $T\mathbf{L} = -\mathbf{L}$, i.e. you get the opposite angular momentum when you apply T.

First of all consider integral spin states. There will be a state with no z-angular momentum, namely $|j, m = 0\rangle$. Applying one T this becomes the same state $|j, m = 0\rangle$ times some phase, but applying T again can only put the state back to exactly $|j, m = 0\rangle$, using the fact that T is antiunitary. So since the phase is the same for all

states that can be superposed, $TT = +1$ for integral spin states.

To understand what happens with half integral spin let us take the simplest example of spin $\frac{1}{2}$. Let us try to fill out our table for just the four special states, up and down along the z-axis, $|+z\rangle$, $|-z\rangle$, and up and down along the x-axis $|+x\rangle$, $|-x\rangle$. Elementary spin theory tells us how these latter two can be expressed in terms of the $|+z\rangle$ and $|-z\rangle$ base states: one of them, $|+x\rangle$, is the in-phase equal superposition, and the other, $|-x\rangle$, is the out-of-phase equal superposition. The physically time reversed state of $|+z\rangle$ is $|-z\rangle$ and vice versa. Likewise, time reversal of $|+x\rangle$ must send us to $|-x\rangle$ within a phase.

For our first entry $T|+z\rangle$ we must have $|-z\rangle$, at least within a phase. This first phase can be chosen arbitrarily, as you can check later, so we may as well take $T|+z\rangle = |-z\rangle$. Now $T|-z\rangle$ must be a phase times $|+z\rangle$. But we cannot choose it to be simply $|+z\rangle$ because then the operation of T on $|+x\rangle$, the in-phase superposition of $|+z\rangle$ and $|-z\rangle$, will only give back the same in-phase state $|+x\rangle$ and not a factor times the out-of-phase state $|-x\rangle$, as it physically must. To make this phase reversal occur we *must* take $T|-z\rangle = -|+z\rangle$, of opposite phase from what we did for $T|+z\rangle$. Now $T(T|+z\rangle) = T|-z\rangle = -|+z\rangle$ and the rest of the table can be filled out. Therefore $TT = -1$ for spin $\frac{1}{2}$, as is easily shown for any half integral spin j, where time reversal never brings us back to the same physical state. Hence combining this with the result for integral spin particles, we have $TT \equiv 360°$ rotation.

Now we come to the sign of the spin $\frac{1}{2}$ loop. You will recall that, with a potential in relativistic quantum mechanics, pairs can be produced so the probability for the vacuum (i.e. the no particle state) to remain the vacuum must be less than one. Write the amplitude for the vacuum remaining the vacuum as $1 + X$, where X is the contribution from all the closed loops drawn on the right hand side of Fig. 6. Then X must contribute a negative amount to the probability for the vacuum to remain the vacuum, which is what the identity in Fig. 6 says because the left hand side is strictly negative.

Consider the loops contributing to X. A loop is constructed by starting with an electron, for example, in a state with Dirac wavefunction u, say, and then propagating around the loop to come back into the same physical state u, and we must take the trace of the resulting matrix product,

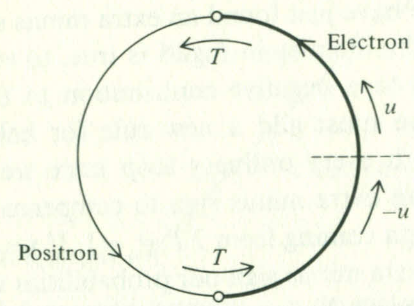

Fig. 12 A particle–antiparticle loop with the two time reversals T shown.

summing over the diagonal elements. But there is a subtlety; the same physical state could have been rotated by 360°, and indeed we see we do have that (or its equivalent, TT). Whatever frame you watch this process from, the electron at some stage changes into a positron moving backwards in time (one T), and then later turns back to an electron moving forwards in time (another T), so propagating round the whole loop you eventually come back to the state TTu; see Fig. 12.

The same TT operator will act in the boson (spin-zero) case too, but there we have $TT = +1$ so there is no problem. In the boson case, everything works out; the ordinary trace in X leads to a negative contribution. But, in the spin $\frac{1}{2}$