Lepton-Nucleus Scattering in a Relativistic Framework: Electromagnetic and Neutral Current Case

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The formalism used to compute the cross-section for electron– and neutrino–nucleus reactions at intermediate energies is surveyed using a relativistic formalism based in the Impulse Aproximation. An extension to incorporate MEC in its fully relativistic form is also outlined.

I. INTRODUCTION

We summarize here the formalism utilized in our calculations [2,3] with the computer codes OUTDW, OUTPW, RFG, MEC3 and some others. We will refer to the case of lepton-nucleus semi-inclusive scattering, and in particular to:

1. Quasielastic neutrino scattering mediated through neutral currents.
2. Quasielastic electron scattering.

We are interested in reactions in which a beam of known energy is used to probe the target nuclei, and a nucleon in the final state is detected and its kinetic energy measured (see fig. 1).

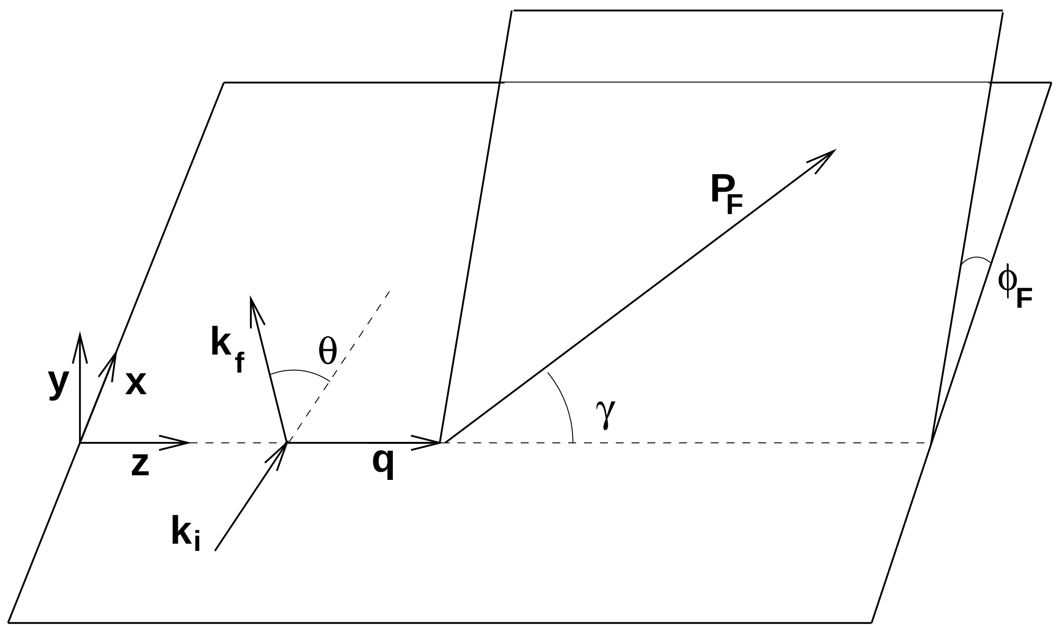

FIG. 1. Kinematics of the exclusive processes considered in this work

A. Conventions

We use the same metric, γ matrices and other conventions as in [18], with *one exception*: the normalization of the free spinors. This also will affect the trace theorems. We use the norm and trace theorems as in [19]. Just to be sure, we use (positive energy) 4-component spinors such that $u^\dagger u = 2E$. The (positive energy) plane waves are of the form:

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$$\psi^{\sigma}_{PW}(\vec{P},\vec{r}) = \frac{1}{\sqrt{2EV}} e^{i\vec{P}\vec{r}} u(\vec{p},\sigma) \tag{1.1}$$

with [1]

$$u(\vec{P},\sigma) = \sqrt{E+M} \begin{pmatrix} \chi^{\sigma} \\ \frac{\vec{\sigma}\vec{P}}{E+M}\chi^{\sigma} \end{pmatrix} \tag{1.2}$$

and χ^{σ} are usual two-component Pauli spinors. We normalize in a box of volume V, such that

$$\int_V d\vec{r} \psi^{\dagger}_{PW} \psi_{PW} = 1$$

.

With respect to the notation, we use the labels as in figures 1 and 2. In particular, the incoming and outgoing lepton momentum are denoted by k_i, k_f, the momentum transfer is q, and the momentum of the detected nucleon is P_F. θ represents the lepton scattering angle, γ is the nucleon angle with respect to $\hat{z} \equiv \hat{q}$ and ϕ_F is its azimuthal angle.

The *missing momentum* is defined as $\vec{P}_m \equiv \vec{P}_I = \vec{q} - \vec{P}_F$. The *missing energy* is the energy converted in internal energy and thus not present (missing) as kinetic energy in the final state:

$$E_m = M_{A-1} - M_A + M \tag{1.3}$$

II. CROSS-SECTION

A. Elementary cross section

In this simplest case, we want to compute $\sigma_{l,l'N}$ elementary, that is, using free nucleons as target.

The invariant matrix element can be written as:

$$\overline{|\mathcal{M}|^2} = g^4 \overline{\sum} |\overline{u} j_{\mu} u \overline{u} J^{\mu} u|^2 \tag{2.1}$$

where g is the coupling constant (so g^2 from each vertex in the squared amplitud). The cross-section can be computed using Fermi's Golden Rule:

$$d\sigma = \frac{1}{2\pi^2} \frac{\overline{|\mathcal{M}|^2}}{2\epsilon_i 2\epsilon_f 2E_I 2E_F} \delta(Energy) d^3\vec{P}_F \tag{2.2}$$

1. Leptonic Tensor

The invariant matrix element can be written as $|\mathcal{M}|^2 = g^4 L_{\mu\nu} H^{\mu\nu}$, where we have introduced the definition of the leptonic and hadronic tensors:

$$L^{\mu\nu} = \overline{\sum_{\sigma_i,\sigma_f}} \overline{u} j^{\mu} u \left[\overline{u} j^{\nu} u\right]^{\dagger} \tag{2.3}$$

and a similar expression for $H^{\mu\nu}$. As it is going to be a common factor in our calculations, we study the leptonic tensor in detail. We use plane wave (PW) for the leptons and the first order Born approximation (PWBA). There are two different cases:

[1] In the codes, the spinors we use are of the same form as in the eq. 1.2, but without the $\sqrt{E+M}$ factor, thus it has to be included in the cross-section at the end

- **Neutral current**

$$
\begin{aligned}
L^Z_{\mu\nu} &= \sum_{\sigma_i,\sigma_f} \overline{u}\gamma_\mu(1\mp\gamma_5)u\,[\overline{u}\gamma_\nu(1\mp\gamma_5)u]^\dagger \\
&= Tr[\gamma_\mu(1\mp\gamma_5)\not{k}_i\gamma_\nu(1\mp\gamma_5)\not{k}_f] \\
&=8\left[\pm i\varepsilon^{\mu\alpha\nu\beta}k_{i\alpha}k_{f\beta}+k_i^\mu k_f^\nu+k_i^\nu k_f^\mu-(k_ik_f)g^{\mu\nu}\right]
\end{aligned}
\tag{2.4}
$$

The *upper* sign corresponds to ν scattering, the lower to $\overline{\nu}$ scattering. We note that we can *add* over initial and final spins for the neutrinoes even though only left-handed neutrinoes exist because the $V \pm A$ term will only give contribution for the adequate case. So there is a factor 2 with respect to the ordinary "sum and average".

If we write the leptonic tensor $L_{\mu\nu} = L^A + L^S$ in terms of the symmetric and antisymmetric part, from eq. 2.4 it can be contracted with a general 4-current $J^\mu = (\rho, \vec{J})$ to get:

$$
\begin{aligned}
L^A_{\mu\nu}J^\mu J^{\nu\dagger} &= \pm 8i \\
&\times\left\{(\vec{k}_i\times\vec{k}_f)2i\Im(\rho^\dagger\vec{J})+(\epsilon_f\vec{k}_i-\epsilon_i\vec{k}_f)(\vec{J}\times\vec{J}^\dagger)\right\}
\end{aligned}
\tag{2.5}
$$

And

$$
L^S_{\mu\nu}J^\mu J^{\nu\dagger} = 4\left\{|(k_i+k_f)^\alpha J_\alpha|^2-|q^\alpha J_\alpha|^2+q_\alpha q^\alpha|J|^2\right\}
\tag{2.6}
$$

Where current conservation has not been used.

- **Electron scattering**

The expressions are very similar to the previous ones, provided we use the ultrarelativistic limit for electrons ($m_e = 0$). If polarizations for the electron are not observed, we get:

$$
\begin{aligned}
L^\gamma_{\mu\nu} &= \frac{1}{2}\sum_{\sigma_i,\sigma_f}\overline{u}\gamma_\mu u\,[\overline{u}\gamma_\nu u]^\dagger \\
&= \frac{1}{2}Tr[\gamma_\mu\not{k}_i\gamma_\nu\not{k}_f] \\
&= 2\left[k_i^\mu k_f^\nu+k_i^\nu k_f^\mu-(k_ik_f)g^{\mu\nu}\right]
\end{aligned}
\tag{2.7}
$$

In this case, $L^\gamma_{\mu\nu}$ is equal to the symmetric part of the neutral current leptonic tensor $L^S_{\mu\nu}$, with a factor $\frac{1}{4}$. If polarizations are measured, then there is no sum and average on spins in eq. 2.7, and the result is identical to the neutral case 2.4, but with a factor $\frac{1}{8}$ and electrons of positive (negative) helicity correspond to the $\overline{\nu}$ (ν) case.

Here we only refer to the unpolarized electron scattering case, as the polarized one is directly and easily related to the neutral current case (all we have to do is to replace the different propagator and coupling constant and to take into account the factor $\frac{1}{8}$).

2. Structure Functions

It is customary to rewrite $L^{\mu\nu}H_{\mu\nu}$ isolating all the dependence on the nuclear current in terms of structure functions and other factors that depend only on the electron kinematics. Using the previous equations and the conventions as in fig. 1, we write for the neutral current:

$$
L_{\mu\nu}J^\mu J^{\nu\dagger} = 4\times 4\epsilon_i\epsilon_f\cos^2(\theta/2)\omega^{\mu\nu}W_{\mu\nu}
\tag{2.8}
$$

where $\omega_{\mu\nu}W^{\mu\nu}$ can be written in terms of a transverse (T), longitudinal (L), and interference (LT,TT,LT',TT') structure functions:

$$\omega_{\mu\nu}W^{\mu\nu} = \omega_L W_L + \omega_T W_T + \omega_{TT} W_{TT} + \omega_{LT} W_{LT} + \omega_{LT'} W_{LT'} + \omega_{TT'} W_{TT'} \quad (2.9)$$

With the following "kinematical" coefficients

$$\omega_L = 1 \quad (2.10)$$
$$\omega_T W_T = \omega_\perp W_\perp + \omega_{||} W_{||} \quad (2.11)$$
$$\omega_\perp = \tan^2(\theta/2) - (1-\cos(2\phi_F))\frac{q_\mu^2}{2\vec{q}^2} \quad (2.12)$$
$$\omega_{||} = \tan^2(\theta/2) - (1+\cos(2\phi_F))\frac{q_\mu^2}{2\vec{q}^2} \quad (2.13)$$
$$\omega_{TT} = \frac{q_\mu^2}{\vec{q}^2} \quad (2.14)$$
$$\omega_{LT} = -2\sqrt{\tan^2(\theta/2) - q_\mu^2/\vec{q}^2} \quad (2.15)$$
$$\omega_{TT'} = \mp 2\tan(\theta/2)\sqrt{\tan^2(\theta/2) - q_\mu^2/\vec{q}^2} \quad (2.16)$$
$$\omega_{LT'} = \pm 2\tan(\theta/2) \quad (2.17)$$

To make explicit the dependence on ϕ_F of the structure functions, a coordinate system is chosen formed by $\hat{n}_\perp$, the direction perpendicular to the nucleon scattering plane (i.e., the plane defined by $\vec{q}$ and $\vec{P}_F$), $\hat{q}$ and $\hat{n}_{||}$, a vector in the nucleon scattering plane perpendicular to $\vec{q}$ and $\hat{n}_\perp$. According to fig. 1. we obtain:

$$\vec{J} = J_q\hat{q} + J_\perp\hat{n}_\perp + J_{||}\hat{n}_{||} \quad (2.18)$$
$$\hat{n}_\perp \equiv (-\sin\phi_F, \cos\phi_F, 0) \quad (2.19)$$
$$\hat{n}_{||} \equiv (\cos\phi_F, \sin\phi_F, 0) \quad (2.20)$$
$$\vec{P}_F \equiv |\vec{P}_F|(\sin\gamma\cos\phi_F, \sin\gamma\sin\phi_F, \cos\gamma) \quad (2.21)$$
$$(2.22)$$

and the relationship between the spatial components of the hadronic current in the xyz and $||\perp z$ reference systems is:

$$J_\perp = -\sin\phi_F J_x + \cos\phi_F J_y \quad (2.23)$$
$$J_{||} = \cos\phi_F J_x + \sin\phi_F J_y \quad (2.24)$$
$$J_x = -\sin\phi_F J_\perp + \cos\phi_F J_{||} \quad (2.25)$$
$$J_y = \cos\phi_F J_\perp + \sin\phi_F J_{||} \quad (2.26)$$

The structure functions can be rewritten as:

$$W_L = |\rho|^2 + \frac{\omega^2}{\vec{q}^2}|J_q|^2 - \frac{\omega}{|\vec{q}|}2\Re(\rho J_q^\dagger) \quad (2.27)$$
$$W_{||} = |J_{||}|^2 \quad (2.28)$$
$$W_\perp = |J_\perp|^2 \quad (2.29)$$
$$W_{TT} = \sin(2\phi_F)\Re(J_\perp J_{||}^\dagger) \quad (2.30)$$
$$W_{LT} = \Re\left\{(\rho - \frac{\omega}{|\vec{q}|}J_q)(-\sin\phi_F J_\perp^\dagger + \cos\phi_F J_{||}^\dagger)\right\} = \Re\left\{(\rho - \frac{\omega}{|\vec{q}|}J_q)J_x^\dagger\right\} \quad (2.31)$$
$$W_{TT'} = \Im(J_{||}J_\perp^\dagger) \quad (2.32)$$
$$W_{LT'} = \Im\left\{(\rho - \frac{\omega}{q}J_q)(\sin\phi_F J_{||}^\dagger + \cos\phi_F J_\perp^\dagger)\right\} = \Im\left\{(\rho - \frac{w}{|\vec{q}|}J_q)J_y^\dagger\right\} \quad (2.33)$$

For the unpolarized electron scattering case, the result is similar but remember that there is a factor $\frac{1}{4}$ in 2.8, and the structure functions coming for the antisymmetric part (TT' and LT') do not contribute.

Current conservation has not been used, so all the spatial components of the current and also the time-like part are present in the expressions.

If the polarization for the final nucleon is measured, some other structure functions will appear [22].

If we use a particular form of the nuclear current, then we can completely determine the structure functions. The simplest case corresponds to the scattering of free nucleons, if we use a *current operator* like[2]:

$$\hat{J}^\mu = F_1\gamma^\mu + F_2\frac{i\sigma^{\mu\alpha}q_\alpha}{2M} + G_1\gamma^\mu\gamma^5 \tag{2.34}$$

Using plane waves for the initial and final nucleon wave functions and summing and averaging on the nucleon spin (*i.e.* $\frac{1}{2}\sum_{\sigma_I,\sigma_F}|\overline{u}(\sigma_F,\vec{P}_F)J^\mu u(\sigma_I,\vec{P}_I)|^2$) we obtain explicit expressions[3]:

$$\begin{aligned}
|\rho|^2 &= G_1^2\left[-4M^2 - \vec{q}^2 + (E_I+E_F)^2\right] && (2.35)\\
&+ \left[F_1^2 - \frac{F_2^2}{4M^2}q_\mu q^\mu\right](E_I+E_F)^2 - \vec{q}^2\left[F_1+F_2\right]^2 &&\\
|J_q|^2 &= G_1^2(4M^2 + 4P_I^zP_F^z - q_\mu q^\mu) - [F_1+F_2]^2(E_I-E_F)^2 + (F_1^2 - q_\mu q^\mu\frac{F_2^2}{4M^2})(2|\vec{P}_F|\cos\gamma - |\vec{q}|)^2 && (2.36)\\
|J_\perp|^2 &= G_1^2\left[4M^2 - q_\mu q^\mu\right] + \left[F_1+F_2\right]^2(-q_\mu q^\mu) && (2.37)\\
|J_{||}|^2 &= |J_\perp|^2 + 4|\vec{P}_F|^2\sin^2\gamma\left[G_1^2 + F_1^2 - \frac{F_2^2}{4M^2}q_\mu q^\mu\right] &&\\
W_{TT} &= 0 && (2.38)\\
2\Re(\rho J_x^\dagger) &= 4|\vec{P}_F|\sin\gamma\cos\phi_F(E_I+E_F) &&\\
&\times\left[G_1^2 + F_1^2 - \frac{F_2^2}{4M^2}q_\mu q^\mu\right] && (2.39)\\
\Im(\rho J_y^\dagger) &= 4(F_1+F_2)G_1|\vec{q}||\vec{P}_F|\sin\gamma\cos\phi_F && (2.40)\\
W_{TT'} &= -4(F_1+F_2)G_1\left\{(E_F-E_I)|\vec{P}_F|\cos\gamma - E_F|\vec{q}|\right\} && (2.41)\\
2\Re(\rho J_q^\dagger) &= 4G_1^2(E_IP_F^z + E_FP_I^z) &&\\
&+ 2(F_1+F_2)^2(E_I-E_F)|\vec{q}| + (F_1^2 - \frac{F_2^2}{4M^2}q_\mu q^\mu)2(E_I+E_F)(2|\vec{P}_F|\cos\gamma - |\vec{q}|) && (2.42)
\end{aligned}$$

In practice, rather than on these expressions we will rely on computing directly the nuclear current J_μ for a particular model (free nucleon, Fermi gas, finite nuclei, DWIA) and obtain the structure functions *numerically* in terms of amplitudes, instead of using trace theorems to obtain *squared* amplitudes.

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[2]An induced pseudoscalar term proportional to q_μ gives no contribution for massless leptons as the ones we consider here

[3]Actually the following expressions are obtained with a cc1 version of the current operator. Both expressions would give identical reusults for *on-shell* nucleons.

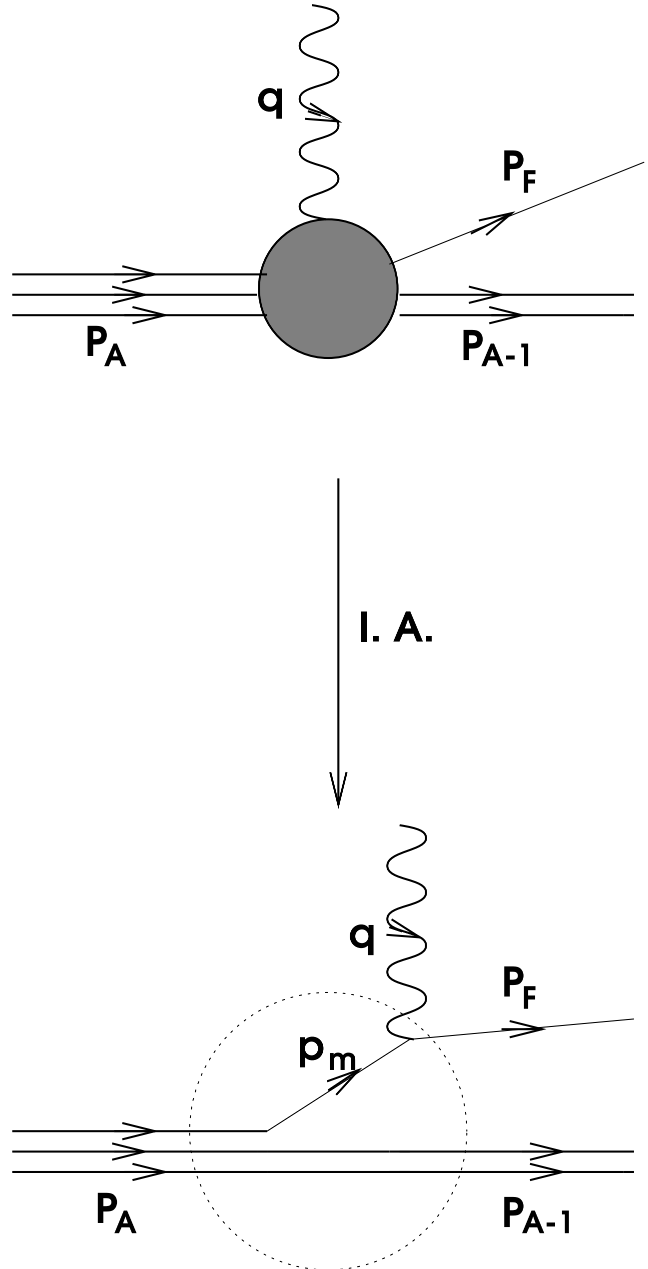

FIG. 2. The $(l, l'N)$ reaction in first order Born Approximation and the Impulse Approximation (IA) picture.

B. Free nucleon cross-section

We start from

$$S_{fi} = ig^2\delta(energy)\int d\vec{x}\int d\vec{y}\int \frac{d\vec{q}}{(2\pi)^2} j^\mu e^{-i\vec{q}(\vec{x}-\vec{y})}\Delta J_\mu(\vec{y}) \tag{2.43}$$

For free nucleons, we use plane waves both for leptons and nucleons:

$$j^\mu = \frac{1}{2V}\frac{1}{(\epsilon_i\epsilon_f)^{1/2}} e^{i(\vec{k}_i-\vec{k}_f)\vec{x}}\overline{u}\hat{j}^\mu u \tag{2.44}$$

$$J^\mu = \frac{1}{2V}\frac{1}{(E_I E_F)^{1/2}} e^{i(\vec{P}_I-\vec{P}_F)\vec{y}}\overline{u}\hat{J}^\mu u \tag{2.45}$$

(note: convention for spinors is $u^\dagger u = 2E$; $\Delta = -1/q_\mu^2$ in the EM case and 1 for weak neutral current in the limit of very heavy vector boson).

The result for the cross-section is:

$$d\sigma = \delta^{(4)}(k_i^\mu - k_f^\mu + P_I^\mu - P_F^\mu)\sigma_M \frac{1}{4}\frac{1}{\epsilon_f^2 E_I E_F} \times \omega^{\mu\nu} W_{\mu\nu} d^3\vec{P}_F d^3\vec{k}_f \tag{2.46}$$

with:

$$\sigma_M^{(e,e'N)} = 4\epsilon_f^2 \cos^2(\theta/2)\frac{1}{q_\mu^4}\left[\frac{g^2}{4\pi}\right]^2 \tag{2.47}$$

$$\sigma_M^{(\nu,\nu' N)} = 4 \times 4\epsilon_f^2 \cos^2(\theta/2)\left[\frac{g^2}{4\pi}\right]^2 \tag{2.48}$$

($g^2 = e^2$ in the EM case and $G_F/\sqrt{2}$ for the neutral current).

We have to integrate on the unobserved final 3-momentum $\vec{k}_f$ o f the neutrino. This is done using the $\delta^{(4)}$ function to give:

$$d\sigma = \delta(\epsilon_i - \epsilon_f + E_I - E_F)\sigma_M \frac{1}{4}\frac{1}{\epsilon_f^2 E_I E_F}\omega^{\mu\nu} W_{\mu\nu} d^3\vec{P}_F \tag{2.49}$$

Now, $d^3\vec{P}_F = E_F|\vec{P}_F|dE_F d\Omega_F$ and $d\Omega_F = d(\cos\gamma)d\phi_F$. The integration on $\cos\gamma$ for the *unobserved* direction of the outgoing nucleon can be done out of the δ of energy that is left. If we allow for some binding energy correction B such that:

$$\begin{aligned} E_I &= (\vec{p}_I^{\,2} + M^2)^{1/2} - B = \overline{E}_I - B \\ &= (\vec{q}^{\,2} + \vec{P}_F^2 - 2|\vec{q}||\vec{P}_F|\cos\gamma + M^2)^{1/2} - B \end{aligned} \tag{2.50}$$

Then:

$$d\sigma = \frac{\overline{E}_I}{|\vec{q}||\vec{P}_F|}\sigma_M \frac{1}{4}\frac{1}{\epsilon_f^2 \overline{E}_I E_F}\omega^{\mu\nu} W_{\mu\nu} E_F|\vec{P}_F|dE_F d\phi_F \tag{2.51}$$

with

$$\cos\gamma = \frac{(E_F + B - \omega)^2 - \vec{q}^{\,2} - \vec{P}_F^2 - M^2}{-2|\vec{q}||\vec{P}_F|} \tag{2.52}$$

the integration on ϕ_F can be done easily, as we have written explicitly the ϕ_F dependence on $W_{\mu\nu}$. This gives a factor 2π for the structure functions that are *independent* of ϕ_F, *i.e.* $W_L, W_T, W_{TT'}$ and the other structure functions vanish. Then, for free nucleons (maybe with some binding energy corrections) we have [4]:

$$\frac{d\sigma}{dE_F} = \frac{\sigma_M}{4\epsilon_f^2}\frac{2\pi}{|\vec{q}|}\left[\omega_L W_L + \omega_T W_T + \omega_{TT'} W_{TT'}\right] \tag{2.53}$$

For the structure functions we can use the expressions for free nucleons as obtained in 2.35, including the sum and average on the nucleon's spin.

C. Relativistic Fermi Gas

In this very simple model, we consider a non-interacting gas of fermions, that is, a initial set of free nucleons with equally distributed initial momentum $|\vec{P}_I| < P_{Fermi}$. To obtain the cross-section we need to average over the initial

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[4]There is a factor $E_I/\overline{E}_I$ with Horowitz *et al.* [5], due to the fact that we are including $\overline{E}_I$ also in the norm of the spinors

nucleon 3-momentum $\int d^3\vec{P}_I$ in 2.48. Numerically, it turns out to be more convenient to make this with the aid of the δ function, so the $\vec{k}_f$ integration is left:

$$d\sigma_{FG} = \delta(\omega + E_I - E_F)\sigma_M \frac{1}{4}\frac{1}{\epsilon_f^2 E_I E_F}\omega^{\mu\nu}W_{\mu\nu}d^3\vec{P}_F d^3\vec{k}_f \tag{2.54}$$

The integrations on $d(\cos\gamma)$ and ϕ_F are performed as before,

$$\begin{aligned} d\sigma_{FG} = & \frac{\sigma_M}{4}\frac{2\pi}{|\vec{q}|}\left[\omega_L W_L + \omega_T W_T + \omega_{TT'}W_{TT'}\right] \\ & \times dE_F d\epsilon_f d(\cos\theta)d\phi \end{aligned} \tag{2.55}$$

As it is always possible to choose the electron scattering plane as the XZ-plane (as in figure 1), the amplitude cannot depend on the scattered electron azimuthal angle ϕ, and after the $d\phi$ integration we just get another 2π factor. Including the $3/(4\pi P_{Fermi}^3)$ normalization average factor for the RFG, we get:

$$\begin{aligned} \frac{d\sigma_{FG}}{dE_F} = & \frac{3\pi}{4P_{Fermi}^3}\int \frac{\sigma_M}{|\vec{q}|}\left[\omega_L W_L + \omega_T W_T + \omega_{TT'}W_{TT'}\right] \\ & \times d\epsilon_f d(\cos\theta) \end{aligned} \tag{2.56}$$

and this expression is *per nucleon* so it needs a factor Z for protons and N for neutrons.

The remaining integrations are going to be performed numerically, so it is convenient to remove the soft (integrable) singularity of the integrand when $|\vec{q}| = 0$, that is a value allowed if $|\vec{P}_F| < P_{Fermi}$ (but it is forbidden if we consider *Pauli blocking*). We just need a change of variable $\beta = |\vec{q}|/(\epsilon_i\epsilon_f)$, then $d\beta = -\dfrac{d\cos\theta}{|\vec{q}|}$ and:

$$\begin{aligned} \frac{d\sigma_{FG}}{dE_F} = & \frac{3\pi}{4\,P_{Fermi}^3}\int_{\beta_{min}}^{\beta_{max}} d\beta \int_{\epsilon_f^{min}}^{\epsilon_f^{max}} d\epsilon_f \sigma_M \\ & \times \left[\omega_L W_L + \omega_T W_T + \omega_{TT'}W_{TT'}\right] \end{aligned} \tag{2.57}$$

with $\beta_{min} = -(\epsilon_i+\epsilon_f)/(\epsilon_i\epsilon_f)$ and $\beta_{max} = -(\epsilon_i-\epsilon_f)/(\epsilon_i\epsilon_f)$. The only tricky point now is to find the proper integration limits for ϵ_f compatible with the kinematical constrains. The range of allowed values is such that:

$$|\vec{P}_I - \vec{P}_F| \leq q \leq |\vec{P}_I + \vec{P}_F| \tag{2.58}$$

$$\epsilon_i - \epsilon_f = \omega \leq q \leq \epsilon_i + \epsilon_f \tag{2.59}$$

That is (note that P_I depends on ϵ_f):

$$|\vec{P}_I - \vec{P}_F| \leq \epsilon_i + \epsilon_f$$

$$\omega \leq |\vec{P}_I + \vec{P}_F|$$

And, of course, there is an ϵ_f^{max} for the minimum value of ω possible, compatible with the kinematics as ω has to be high enough to account for the kinetic energy of the ejected nucleon and the missing energy.

D. General Case

We start from the usual transition matrix element, and we factorize the temporal dependence of the currents, recalling 2.43:

$$S_{fi} = ig^2\delta(energy)\int d\vec{x}\int d\vec{y}\int \frac{d\vec{q}}{(2\pi)^2} j^\mu e^{-i\vec{q}(\vec{x}-\vec{y})}\Delta J_\mu(\vec{y}) \tag{2.60}$$

As before, $\Delta = -1/q_\mu^2$ for the EM case and 1 for the weak neutral current.

Using plane waves for the leptons, the current j^μ is expressed in the usual way:

$$j^\mu = \frac{1}{2V}\frac{1}{(\epsilon_i\epsilon_f)^{1/2}}e^{i(\vec{k}_i-\vec{k}_f)\vec{x}}\overline{u}\hat{j}^\mu u \tag{2.61}$$

introducing this into 2.60 we can perform some of the integrations and we get:

$$S_{fi} = ig^2\delta(energy)\frac{\pi}{V\sqrt{\epsilon_i\epsilon_f}}\overline{u}\hat{j}^\mu u\Delta\int d\vec{y}e^{i\vec{q}\vec{y}}J^N_\mu(\vec{y}) \tag{2.62}$$

with $\vec{q} = \vec{k}_i - \vec{k}_f$.

The nuclear current matrix element J^μ_N for the process depicted in figure 2 is computed as

$$J^\mu_N =< \psi^{(-)}(P_F, \sigma_F, A-1, P_{A-1})|\hat{J}|\psi_A(P_A) > \tag{2.63}$$

That is, between an intrinsic initial A-body nuclear ground state $\psi_A(P_A)$ and a final scattering state with the boundary conditions of having a final nucleon scattered with momentum P_F and spin σ_F.

To compute this matrix element, it is customary to use a number of approximations that define the "Distorted Wave Impulse Approximation" (DWIA):

1. For the initial and final wave function a shell model approach is used. Formally this is achieved by writing 2.63 in terms of an independent particle description (see for example Ref. [17]) and using an *effective* current operator:

$$J^\mu_N =< P_F, \sigma_F, A-1, P'_{A-1}|\hat{J}^\mu_{eff}|\psi^{SM}_B, A-1, P_{A-1} > \tag{2.64}$$

   where $|P_F, \sigma_F, A-1, P_{A-1} >$ is a single-channel optical-model wave function constructed from the product of a final state for the A-1 particles residual nucleus and either a plane wave (PWIA) or distorted wave (DWIA) description of the outgoing ejected nucleon. $|\psi^{SM}_B, A-1, P_{A-1} >$ denotes a single-particle shell model wave function coupled to the rest of the initial nucleus. Now, all the complexities inherent to the use of *exact* wave functions have been incorporated in the unknown effective current operator, that in general should be a rather complicated many-body operator.

2. In the IA, the effective current operator is substituted for the *free* one-body current operator. Of course this is an extreme approximation that in general is not justified. By doing that, current conservation at the nuclear vertex cannot be guaranteed. Furthermore, due to the off-shell nature of the nucleons involved in the process, there is some ambiguity in the expression for the free current operator used and the treatment of its off-shell character.

Even though, for the kinematical conditions that are used in $(e, e'N)$ or $(\nu, \nu'N)$, the IA is expected to be a fair approximation [13].

Performing the angular momentum algebra involved in the shell model description for the residual and target nuclei, the nuclear current can be expressed in terms of spectroscopic amplitudes $A(j, A-1, \cdots)$ times single-particle current matrix elements. Furthermore it is assumed that the detected nucleon can originate only in one particular shell of the target nucleus, determined by energy conservation and the kinematics of the experiment. The required single-particle matrix elements are of the form:

$$J^{N\mu}(\vec{r}) = \overline{\psi}^{\sigma_F}(\vec{r})\hat{J}^\mu_N\psi^{jm}_B(\vec{r}) \; , \tag{2.65}$$

where $\psi^{jm}_B, \psi^{\sigma_F}$ are the wave functions for the initial bound nucleon and final nucleon, respectively, and $\hat{J}^\mu_N$ is the single-particle current operator for free nucleons, to be specified later.

Now the cross-section can be calculated from $\overline{|S_{fi}|^2}/(VT)$, dividing by the incident flux and multiplying by the final state density. If we take into account the presence of the residual nucleus, the density of final states is: $V\frac{d\vec{k}_f}{(2\pi)^3}$, $V\frac{d\vec{P}_f}{(2\pi)^3}$ for the final lepton and nucleon, and $V\frac{d\vec{P}_{A-1}}{(2\pi)^3}\rho(M_{A-1})dM_{A-1}$ for the residual nucleus, where we have taken into account that its energy and 3-momentum are not correlated. After factorizing the motion of the residual nucleus and the ejected nucleon we get:

$$\begin{aligned} d\sigma = {} & g^4\delta^{(4)}(q^\mu + P^\mu_A - P^\mu_{A-1} - P^\mu_F)\Delta^2\frac{1}{(2\pi)^5}\frac{|A(j, A-1, \cdots)|^2}{2\epsilon_i 2\epsilon_f} \\ & \times \overline{|\overline{u}j_\mu u J^\mu_N(\vec{q})|^2}d\vec{k}_f d\vec{P}_F d\vec{P}_{A-1}\rho(M_{A-1})dM_{A-1} \end{aligned} \tag{2.66}$$

Where we have indicated that some sum and/or average on spins has to be performed, and $J^{N\mu}(\vec{q})$ is given by[5]

$$J^{N\mu}(\vec{q}) = \sqrt{V}\int d\vec{r}e^{i\vec{q}\vec{r}}\overline{\psi}^{\sigma_F}(\vec{P}_F, E_F, \vec{r})\hat{J}^{\mu}\psi_B^{jm}(\vec{r}) \tag{2.67}$$

After integration on the non-measured variables $\vec{P}_{A-1}$, M_{A-1} the result is:

$$\begin{aligned} d\sigma = & g^4\frac{E_{A-1}}{M_{A-1}}\Delta^2\frac{1}{(2\pi)^5}\frac{|A(j,A-1)|^2}{2\epsilon_i 2\epsilon_f}\overline{|\overline{u}j_\mu u J_N^\mu(\vec{q})|^2} \\ & \times \rho(M_{A-1})\epsilon_f^2 d\epsilon_f d\Omega_f|\vec{P}_F|E_F dE_F d\Omega_F \end{aligned} \tag{2.68}$$

with the following constrains:

$$E_{A-1} = E_A + \omega - E_F \tag{2.69}$$

$$M_{A-1} = \sqrt{E_{A-1}^2 - \vec{P}_{A-1}^2} \tag{2.70}$$

$$\vec{P}_{A-1} = \vec{P}_A + \vec{q} - \vec{P}_F \tag{2.71}$$

E. Relativistic Bound State

Following more or less the same procedure as for the RFG, we can now compute the case in which the initial nucleon is not free but instead is represented by a bound state wave function. We use the expression for the nuclear current in *coordinate space* and a plane wave for the outgoing nucleon. This will give for 2.67:

$$J^{N\mu}(\vec{q}) = \frac{1}{\sqrt{2E_F}}\overline{u}(\vec{P}_F, \sigma_F)\int d\vec{r}e^{-i\vec{P}_I\vec{r}}\hat{J}\psi_B^{jm}(\vec{r}) \tag{2.72}$$

with $\vec{P}_I = \vec{P}_F - \vec{q}$. If $\hat{J}$ does not depend on $\vec{r}$ (and this is more or less true in PWBA and using operators in the *cc2* category of De Forest [7]), this can be written as:

$$J^{N\mu}(\vec{q}) = \frac{1}{\sqrt{2E_F}}\overline{u}\hat{J}\psi_B^{jm}(\vec{P}_I) \tag{2.73}$$

That is, the nuclear matrix element can be written in terms of the wave function for the bound nucleon in momentum space ($\vec{P}_I = \vec{P}_F - \vec{q}$). We are going to use 4-component spinors, solutions of the Dirac equation with good total angular momentum quantum numbers j, m, and that in configuration space are given by:

$$\psi_B^{jm}(\vec{r}) \equiv \psi_\kappa^m(\vec{r}) = \begin{pmatrix} \mathrm{u}_\kappa(r)\phi_\kappa^m(\hat{r}) \\ \mathrm{i}\ \mathrm{w}_\kappa(r)\phi_{-\kappa}^m(\hat{r}) \end{pmatrix} \tag{2.74}$$

eigenstates of total angular momentum with quantum numbers κ, m ($j = |\kappa| - 1/2$; $l = \kappa$ if $\kappa > 0$ and $l = -\kappa - 1$ if $\kappa < 0$. It is also introduced $\bar{l} = l - \frac{|\kappa|}{\kappa}$). The functions u_κ, w_κ satisfy the usual radial equations [10] and $\phi_\kappa^m(\hat{r})$ is given by

$$\phi_\kappa^m(\hat{r}) = [Y_l \otimes \vec{\sigma}]_j^m \equiv \sum_{m',\sigma} < l\ m'\ \frac{1}{2}\ \sigma |\ j\ m > Y_{lm'}(\hat{r})\chi_\sigma^{\frac{1}{2}} \tag{2.75}$$

The normalization we use is $\int_V \psi_\kappa^{jm\dagger}(\vec{r})\psi_\kappa^{jm}(\vec{r})d\vec{r} = 1$. For instance, the solutions obtained with the code TIMORA [15] through a Hartree procedure are of this kind.

In momentum space we have:

---

[5]Note that due to the CDM motion some small corrections should be incorporated in $\vec{q}$.

$$\psi_\kappa^m(\vec{P}_I) = \int d\vec{r} e^{-i\vec{P}_I\vec{r}}\psi_\kappa^m(\vec{r})$$
$$= 4\pi(-i)^l \begin{pmatrix} \text{u}_\kappa(P_I)\phi_\kappa^m(\hat{P}_I) \\ \text{-s}_\kappa w_\kappa(P_I)\phi_{-\kappa}^m(\hat{P}_I) \end{pmatrix} \tag{2.76}$$

here $s_\kappa = \kappa/|\kappa|$ and $u_\kappa(P) = \int r^2 dr u(r) j_l(Pr)$, $w_\kappa(P) = \int r^2 dr w(r) j_{\bar{l}}(Pr)$.
If we substitute this in the expression of the cross-section 2.68, we have:

$$d\sigma^{PWIA} = g^4 \frac{\epsilon_f}{\epsilon_i}|\vec{P}_F||A(j,A-1)|^2\rho(M_{A-1})\frac{1}{(2\pi)^3}\frac{E_{A-1}}{M_{A-1}}\frac{1}{2}$$
$$\times\, \Delta^2\overline{|j_\mu J_{PW}^{N\mu}(\vec{q})|^2} dE_F d\epsilon_f d\Omega_F d\Omega_f \tag{2.77}$$

where $J_{PW}^{N\mu} = \dfrac{\overline{u}\hat{J}^\mu\psi_\kappa^m(\vec{P}_I)}{4\pi}$. Recalling 2.8 and the definition of σ_M we get:

$$\overline{|j_\mu J^\mu|^2} = \sigma_M \frac{\epsilon_i}{\epsilon_f}\left[\frac{4\pi}{g^2}\right]^2 \omega^{\mu\nu}\overline{W}_{\mu\nu} \tag{2.78}$$

and the final result is:

$$d\sigma^{PWIA} = |\vec{P}_F||A(j,A-1,\cdots)|^2\rho(M_{A-1})\frac{\sigma_M}{\pi}\frac{E_{A-1}}{M_{A-1}}$$
$$\times\, \omega_{\mu\nu}\overline{W}_{PW}^{\mu\nu} dE_F d\epsilon_f d\Omega_F d\Omega_f \tag{2.79}$$

Note that the sum and/or average on the lepton's spin is already included in the expressions for $\omega^{\mu\nu}$ and σ_M. The remaining sum and average on the *nucleon's* spin is indicated, and we should to understand for example:

$$\overline{W}_L = \frac{1}{2j+1}\sum_{\sigma_F,m} W_L \tag{2.80}$$

To obtain W_L the expression for $J_{PW}^{N\mu}$ is used. As before, the integration on ϕ_F can be performed easily and we get only contributions from the $W_L, W_T, W_{TT'}$ structure functions and a factor 2π. On γ we integrate using:

$$\rho(M_{A-1}) = \delta(M_{A-1} - M_{A-1}^0 - \varepsilon_j) = \delta(\sqrt{E_{A-1}^2 - \vec{P}_{A-1}^2} - M_{A-1}^0 - \varepsilon_j) \tag{2.81}$$
$$= \delta(\sqrt{(M_A + \omega - E_F)^2 - (\vec{q} - \vec{P}_F)^2} - M_{A-1}^0 - \varepsilon_j)$$

We see here that the missing energy appears in the density of states for the residual nucleus: $M_{A-1} - M_{A-1}^0$. It can be fixed through the kinematics (see 2.69—2.71), while $M_{A-1}^0 + \varepsilon_j$ is the mass of the residual nucleus for the shell we are interested in.
Using all this in 2.79, we get a factor $\dfrac{M_{A-1}}{|\vec{q}||\vec{P}_F|}$ and the condition:

$$\cos\gamma = \frac{M_{A-1}^2 + \vec{q}^2 + \vec{P}_F^2 - E_{A-1}^2}{2|\vec{q}||\vec{P}_F|} \tag{2.82}$$

Finally

$$\frac{d\sigma^{PWIA}}{dE_F d\epsilon_f d\Omega_f} = |A(j,A-1,\cdots)|^2\frac{2\sigma_M}{|q|}E_{A-1}$$
$$\times\left[\omega_L\overline{W}_L + \omega_T\overline{W}_T + \omega_{TT'}\overline{W}_{TT'}\right] \tag{2.83}$$

The integration on ϕ is trivial as before, and we get:

$$\frac{d\sigma^{PWIA}}{dE_F} = 4\pi|A(j,A-1,\cdots)|^2\int d\epsilon_f\int d(\cos\theta)E_{A-1}\frac{\sigma_M}{|\vec{q}|}$$
$$\times\left[\omega_L\overline{W}_L + \omega_T\overline{W}_T + \omega_{TT'}\overline{W}_{TT'}\right] \tag{2.84}$$

For closed shell nuclei in the extreme shell model, we have $|A(j, A-1, \cdots)|^2 = 2j+1$. If several shells can contribute (as for example if the missing energy is not determined because we do not observe the scattered lepton) we just sum the corresponding cross-section for every shell. (note: a change of variable to remove the $1/|\vec{q}|$ term is also convenient).

Again, the tricky part has to do with the integration limits for ϵ_f. One should keep in mind that:

$$\omega_{min} = M_{A-1} + E_F - M_A \tag{2.85}$$

and the kinematical constrains:

$$|\vec{P}_{A-1} - \vec{P}_F| \leq |\vec{q}| \leq |\vec{P}_{A-1} + \vec{P}_F| \tag{2.86}$$
$$\epsilon_i - \epsilon_f = \omega \leq |\vec{q}| \leq \epsilon_i + \epsilon_f \tag{2.87}$$

F. PWIA and factorization

In the nonrelativistic formalism and using PWIA for the ejected nucleon, a factorization prescription holds for the cross-sections we are computing, so that it is possible to write them as the product of a kinematical factor, a nuclear spectral function $S(E_m, \vec{P}_I)$ independent of the reaction (that essentially gives the *probability* of having a nucleon in the target nucleus with determined missing energy and momentum) and an elementary $\sigma_{l,l'N}$ cross-section as the one given in 2.53, that can be computed without any nuclear information. It may be interesting to see if this factorization prescription also holds in the relativistic case.

In general, $\sigma_{l,l'N}$ is computed starting from $(\overline{u}\hat{J}^N_\mu u)$, and using *free spinors* . These ones are *positive* energy solutions of the free Dirac equation and their upper and lower components are related by:

$$\frac{\vec{\sigma}\vec{P}}{E+M}\phi^{up} = \phi^{down} \tag{2.88}$$

In the nonrelativistic case, usually one starts from a relativistic hamiltonian that is computed between positive energy solutions that verify 2.88 and some kind of nonrelativistic expansion is performed afterwards.

On the other hand, in the relativistic case for *nonfree* solutions, a certain amount of coupling to the nonnegative energy plane exists or, in other words, eq. 2.88 is not verified. This will cause a departure from the behavior of the elementary cross-section (for example in the extraction of the structure functions) and also a difference with the nonrelativistic case, restricted only to positive energy solutions. Thus in the relativistic case, even in PWIA where for the final nucleon we use plane waves that indeeed verify 2.88, as however 2.88 is not verified by the initial bound state wave function, factorization is not *exactly* recovered.

We can project the relativistic wave functions on the positive energy plane by imposing eq. 2.88. In this way, in PWIA we expect to recover factorization.

The resulting reduced cross-section defined as:

$$\rho(P_I) = \int_{\Delta E_m} \frac{d\sigma}{dE_F\epsilon_f\Omega_F\Omega_f}\frac{1}{\sigma_{l,l'N}K}dE_m \tag{2.89}$$

is proportional to the momentum distribution. In other words, if factorization and a shell model approach as it was described is assumed, the spectral function is expected to be of the form:

$$\delta(M_{A-1} - M^0_{A-1} - \varepsilon_j)|A(j, A-1, \cdots)|^2 \frac{1}{4\pi}\overline{|\psi(\vec{P}_I)|}^2$$

where[6]

$$\overline{|\psi(\vec{P}_I)|}^2 = \int d\Omega \psi^\dagger(\vec{P}_I)\psi(\vec{P}_I) = \frac{2}{\pi}[|u(P_I)|^2 + |w(P_I)|^2] \tag{2.90}$$

$$\psi(\vec{P}_I) = \frac{1}{(2\pi)^{3/2}}\int d\vec{r} e^{-i\vec{P}_I\vec{r}}\psi_B^{jm}(\vec{r}) \tag{2.91}$$

And then in eq. 2.89 the reduced cross-section is proportional to the momentum distribution. In this case (PWIA), if factorization is verified, as the momemtum distribution only depends on $|\vec{P}_I|$ the reduced cross section is symmetric with respect to the $|\vec{P}_I| = 0$ value.

[6]Note that there is a factor $1/(2\pi)^{3/2}$ in the momentum space expression of ψ 2.91 with respect to 2.76.

G. DWIA

In this case we substitute the value of $\psi_F(\vec{P}_F, E_F, \vec{r})$ in 2.67 by a partial wave expansion:

$$\psi^{\sigma_F}(\vec{r}) = 4\pi\sqrt{\frac{E_F+M}{2E_F V}}\sum_{\kappa,\mu,m} e^{-i\delta^*_\kappa} i^l < l\ m\ \frac{1}{2}\ \sigma_F | j\ \mu > \\ \times Y^*_{lm}(\hat{P}_F)\psi^\mu_\kappa(\vec{r})\ , \tag{2.92}$$

The potential may be in general complex (as it is the case for phenomenological scalar-vector optical potentials), so the phase-shifts and radial functions are also complex and since the wave function corresponds to an outgoing proton we have to use in Eq. (2.92) the complex conjugates f^*_κ, g^*_κ (equivalent to w, u in 2.74) and the complex conjugate phase-shift δ^* with a negative sign.

If we define the nuclear current matrix element in DWIA as

$$J^{N\mu}_{DW} = \sqrt{\frac{2E_F}{E_F+M}}\frac{1}{4\pi}\sqrt{V}\int d\vec{r} e^{i\vec{q}\vec{r}}\overline{\psi}^{\sigma_F}(\vec{r})\hat{J}^\mu\psi^{jm}_B(\vec{r}) \tag{2.93}$$

Then the result in DWIA is:

$$d\sigma^{DWIA} = |\vec{P}_F||A(j, A-1)|^2(E_F+M)\rho(M_{A-1})\frac{\sigma_M}{\pi}\frac{E_{A-1}}{M_{A-1}} \\ \times\ \omega_{\mu\nu}\overline{W}^{\mu\nu} dE_F d\epsilon_f d\Omega_F d\Omega_f \tag{2.94}$$

Following exactly the same steps as for the PWIA case we get the result:

$$\frac{d\sigma^{DWIA}}{dE_F} = (E_F+M)|A(j, A-1, \cdots)|^2(4\pi)\int d\epsilon_f\int d(\cos\theta)E_{A-1}\frac{\sigma_M}{|\vec{q}|} \\ \times\ \left[\omega_L\overline{W}_L + \omega_T\overline{W}_T + \omega_{TT'}\overline{W}_{TT'}\right] \tag{2.95}$$

As for the PWIA case, $W_L, W_T, W_{TT'}$ are evaluated numerically from every component of the nuclear matrix element.

1. *Factorization*

In general, in DWIA factorization is not exactly recovered. In the nonrelativistic approach this is due to the spin-orbit term in the optical potential. In the relativistic case it is due to the same reason that in PWIA, but now this *also* applies to the ejected nucleon. It is possible to define a *distorted spectral function* by analogy to the one defined in PWIA, obtained by substituting the momentum space expression for the bound nucleon wave function by a *distorted* version:

$$\psi_{DW}(\vec{P}_F, \vec{q}) = \frac{1}{(2\pi)^{3/2}}\int \psi^{\sigma_F\dagger}(E_F, \vec{P}_F, \vec{r})e^{-i\vec{q}\vec{r}}\psi^{jm}_B(\vec{r})d\vec{r} \tag{2.96}$$

The distorted momentum distribution is given by this expression after sum an average on the nucleon's spins.

Contrary to the PWIA case, this distorted momentum distribution is not symmetric around $|\vec{P}_I| = 0$ in general, as it depends not only on $\vec{P}_I = \vec{P}_F - \vec{q}$ but rather on *both* $\vec{P}_F$ and $\vec{q}$.

III. MEC

In the previous pages the assumption of the nuclear current being a one body operator has been made. Here we want extend our study to the case that a two body part is also present.

To reach a final two particles two holes ($2p2h$) state starting from a mean field $0p0h$ state we will need to go beyond the Impulse Approximation, in which only one-body current, and one exchanged vector boson plays a role, and the interaction is described in terms of a single nucleon. So to explain reactions as $(e, e'2N)$ one needs either to include correlations in the ground state, thus a simple shell model $0p0h$ wave function for the target nuclei is not exact, or to take into account the possible emission of more than one ejectile through rescattering due to the Final State Interaction, or two body components in the current.

As seen in the preceeding discussion (eq. 2.65), one can rearrange things in order to include the complexities related to the "exact" nature of the wave function in the current operator. In what follows we will assume that the target nucleus is well described by a $0p0h$ state, and study the matrix elements of one-body and two-body currents (see figure 3).

A. Two-body currents and many-body formalism: Second Quantization

The general expression for any two-body operator in second quantization is [21]:

$$\hat{J} = \frac{1}{2} \sum_{p_1, p'_1, p_2, p'_2} a^+_{p'_1} a^+_{p'_2} a_{p_2} a_{p_1} < p_{1'} p_{2'} | \hat{J} | p_1 p_2 > \tag{3.1}$$

where $p_1 \neq p_2$ as we are considering here only the 2-body part.

1. 1p1h final state

In the quasielastic region this is the kind of excited states that are expected to contribute the most. In the case of $(e, e'p)$ reactions, as the final state is well determined, one can work below the treshold for knocking out more than one nucleon, so it is known for sure that only $1p1h$ states are excited with respect to the target nuclei.

For neutrino scattering, as the scattered neutrino is not detected, it is not well determined the transferred energy to the nucleus and the missing energy involved in each process is not known. But in this case, by imposing kinematical constrains on the final nucleon, one can reduce the phase space available for such processes. That is, as the kinetic energy that can be transferred to the nuclear system is rather limited (as the energy of the initial neutrino is in principle well determined) we can set a cutoff at some minimum value of the kinetic energy for the detected nucleon, in such a way that the phase space available for the competing events in which more than one nucleon is emited (the missing energy needed would be greater by the amount of the additional extraction energy, about 14-18 MeV for ^{12}C) is greatly reduced.

We are interested in $1p1h$ final state, in which a nucleon of the target nucleus in state $|P_I>$ is knocked-out and detected in state $|P_F>$:

$$|P_F P_I> = \theta(|\vec{P}_F| - P_{Fermi}) \theta(P_{Fermi} - |\vec{P}_I|) a^+_{P_F} a_{P_I} |F> \tag{3.2}$$

where $|F>$ represents a Fermi-vacuum ground state and θ is a step function.

If $< P_F P_I | \hat{J} | F >$ is computed using 3.1 and $|F>$ the many-body Fermi vacuum, four contributions are obtained [21,22]. Note that antisymmetrization is implicitly included in the second quantization formalism, so all the crossed diagrams contributions are already in there, and also the minus sign for fermionic loops is recovered in the resulting expression:

$$\begin{aligned} < P_F P_I | \hat{J} | F > &= \frac{1}{2} \theta(|\vec{P}_F| - P_{Fermi}) \theta(P_{Fermi} - |\vec{P}_I|) \\ &\times \sum_p \theta(P_{Fermi} - |\vec{p}|) \left[J(p,p,P_I,P_F) + J(P_I,P_F,p,p) - J(P_I,p,p,P_F) - J(p,P_F,P_I,p) \right] \end{aligned} \tag{3.3}$$

In our case, $|P_F>$ is the detected nucleon (asymptotically free), and $|P_I>$ is the nucleon removed from the target nucleus. (see fig. 2).

Furthermore, taken into account the symmetry of the matrix elements $J(1, 1', 2, 2') \equiv J(2, 2', 1, 1')$ we can rearrange the previous expression. Also, it is possible to show quite generally that the "direct" terms are zero. The only non zero contribution is given by the "crossed" terms (see figure 5).

The result is then:

$$< P_F P_I | \hat{J} | F > = -\theta(|\vec{P}|_F - P_{Fermi}) \theta(P_{Fermi} - |\vec{P}_I|) \sum_p \theta(P_{Fermi} - |\vec{p}|) J(P_I, p, p, P_F) \tag{3.4}$$

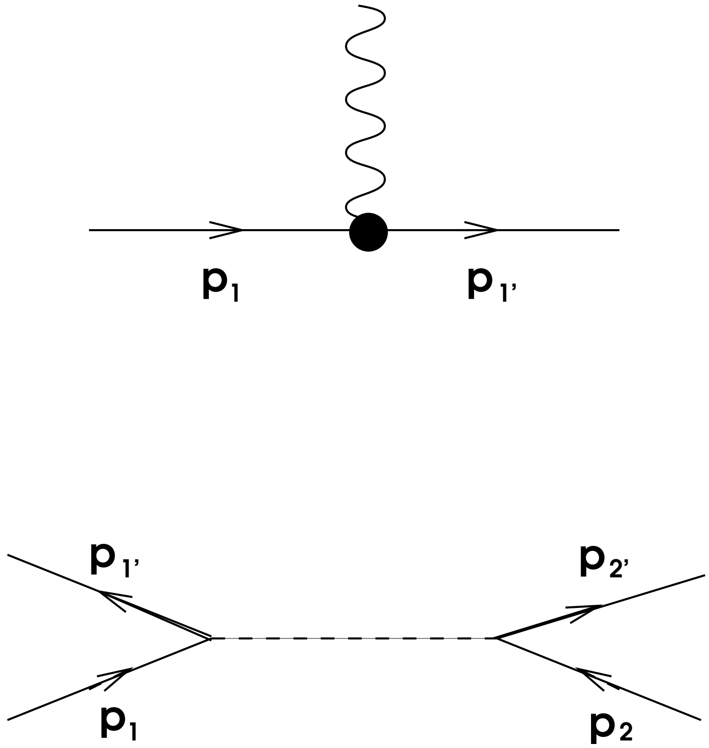

FIG. 3. Examples of one-body $J(p_1, p_1')$ (upper) and two-body $J(p_1, p_1', p_2, p_2')$ (lower) currents

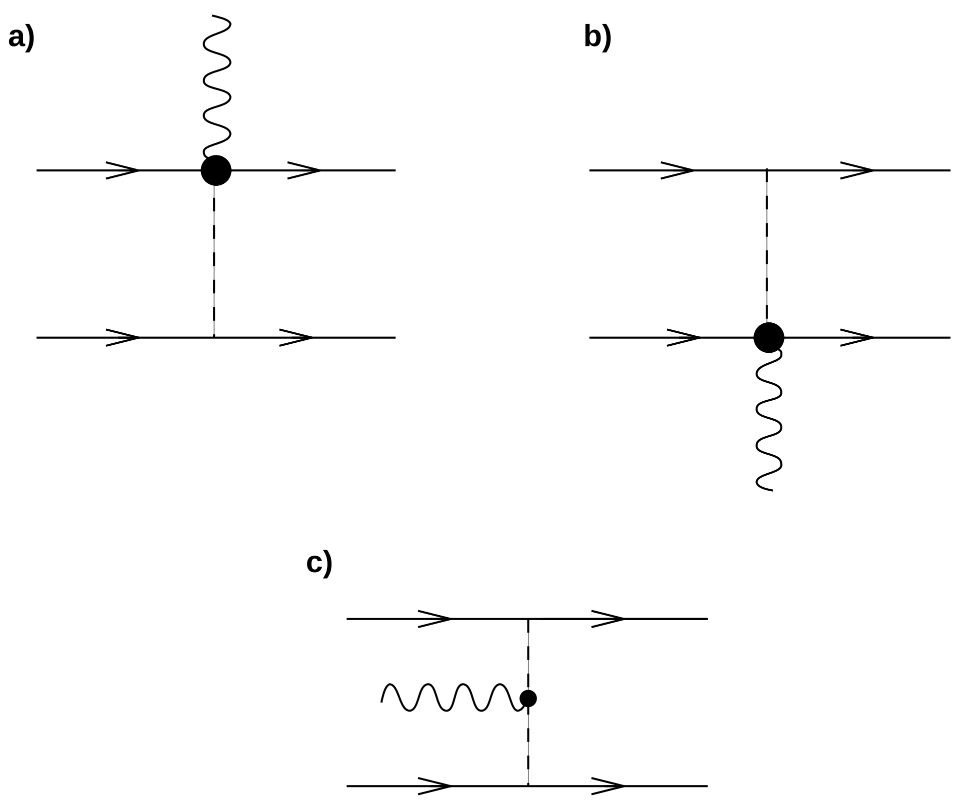

FIG. 4. Some 2-body currents due to one pion exchange. Diagram c) represents the "pion-in-flight" contribution. Diagrams a) and b) "pair currents".

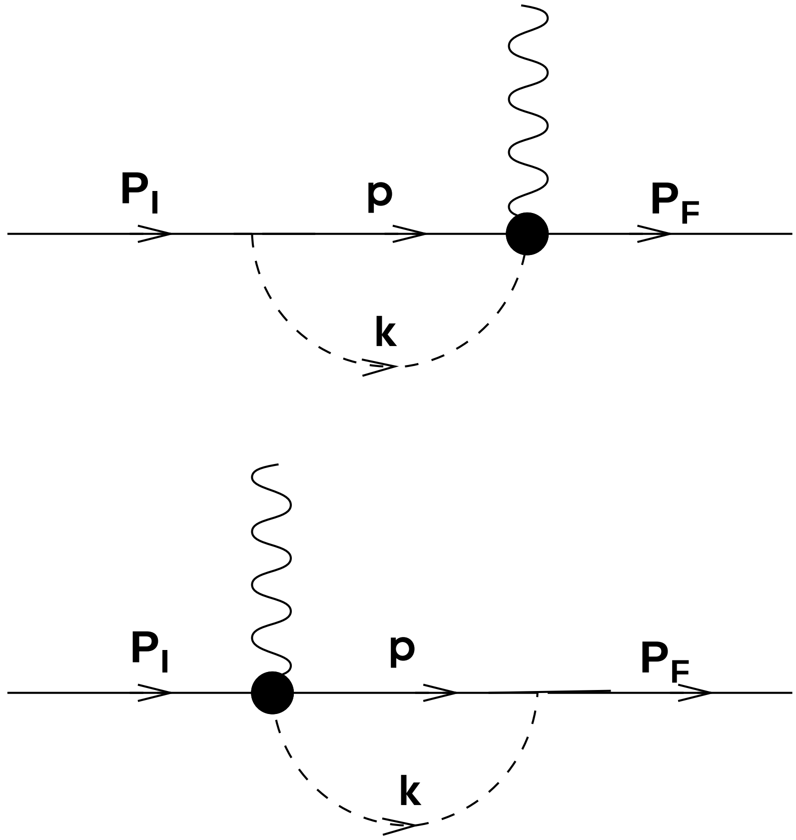

FIG. 5. Contribution of diagrams a) (upper part) and b) (lower part) of figure 4 to the $< 1p1h|J^{(2)}|F >$ matrix element.

B. Cross-section

We start writing the transition matrix element between the initial state $|F >$ (Fermi vacuum) and the $|1p1h >$ final state mediated through the one-body and the two-body current:

$$S_{fi} = \sum_i < P_F P_I|\hat{J}_i^{(1)}|F > + \sum_{i \neq j} < P_F P_I|\hat{J}_{i,j}^{(2)}|F > \tag{3.5}$$

As seen before, using the second quantization language, we can write the one-body and two-body parts as:

$$\hat{J}^{(1)} = \sum_{p_1, p_{1'}} J(p_1, p_{1'}) a^+_{p_{1'}} a_{p_1} \tag{3.6}$$

$$\hat{J}^{(2)} = \frac{1}{2} \sum_{p_1, p'_1, p_2, p'_2} a^+_{p'_1} a^+_{p'_2} a_{p_2} a_{p_1} J(p_1, p_{1'}, p_2, p_{2'}) \tag{3.7}$$

in this last equation it is understood that $p_1 \neq p_2, p_{1'} \neq p_{2'}$. For $J(p_1, p_{1'})$ we can use the IA expressions (fig. 3a):

$$J_{IA}(p_1, p_{1'}) = ig^2 \int d^4x \int d^4y \int \frac{d^4q}{(2\pi)^4} j^\mu e^{iq^\mu(x-y)_\mu} J^{IA}_\mu(y) \Delta \tag{3.8}$$

where Δ represents the vector-boson propagator. For the two body part $J(p_1, p_{1'}, p_2, p_{2'})$ we compute for instance the "pair" diagrams for one pion exchange (diagrams 4-a, 4-b):

$$J_{OPE}(p_1, p_{1'}, p_2, p_{2'}) = ig^2 \int d^4x \int d^4y \int \frac{d^4q}{(2\pi)^4} j^\mu e^{iq^\mu(x-y)_\mu} J^{\pi(+)}_\mu(y) \Delta \int d^4z G_\pi(y,z) J^{(-)}_\pi(z) + (1, 1' \Leftrightarrow 2, 2') \tag{3.9}$$

where the $(-)$ superscript refers to the absorption of a pion, and the $(+)$ to production induced by the external current.

As the expression of the current is now rather complicated, we are going to use a simple nuclear model. This is a complementary situation to the one found previously where the Impulse Approximation allowed us to deal rather simply with relatively complicated nuclear models (including FSI). Here we restrict ourselves to the RFG.

In this model we can compute the expressions 3.8, 3.9 using plane waves both for leptons and nucleons, including an isospin part. The result is:

$$J_{IA}^{PW}(p_1,p_1) = N_i N_f N_1 N_{1'} \delta^{(4)}(k_i^\mu + p_1^\mu - p_{1'}^\mu - k_f^\mu) i g^2 (2\pi)^4 [\overline{u}_f \hat{j}^\mu u_i] \ \Delta \ [\overline{u}_{1'} \hat{J}_\mu^{IA} u_1] \chi_{\tau_{1'}}^\dagger \mathbf{1} \chi_{\tau_1} \tag{3.10}$$

where the N's are normalizations factors (see eqs. 1.1 and 1.2). For eq. 3.9 we write:

$$J_{OPE}^{PW}(p_1,p_{1'},p_2,p_{2'}) = N_i N_f N_1 N_{1'} N_2 N_{2'} \delta^{(4)}(k_i^\mu + p_1^\mu - p_{1'}^\mu + p_2^\mu - p_{2'}^\mu - k_f^\mu) i g^2 (2\pi)^4 \times \tag{3.11}$$

$$[\overline{u}_f \hat{j}^\mu u_i] \Delta [\overline{u}_{1'} \hat{J}_\mu^{\pi(+)} u_1] \chi_{\tau_{1'}}^\dagger \hat{O}_{\pi(+)} \chi_{\tau_1} \frac{i d^\pi(q)}{k_\mu^2 - m_\pi^2} \overline{u}_{2'} \hat{J}^{\pi(-)} u_2 \chi_{\tau_{2'}}^\dagger \hat{O}_{\pi(-)} \chi_{\tau_2} + (1,1' \Leftrightarrow 2,2')$$

where $d^\pi(q)$ represents the pion form factor that we set to unity in what follows. Provided that, as well as we know from the previous sections the expressions for the leptonic current j_μ (eqs. 2.4, 2.7) and the nuclear current operator J_μ^{IA} (eq. 2.34), we also have the expressions for the operator for absorption and production of a pion, we can compute the previous matrix elements.[7]

Expression 3.5 can be worked out easily in the case of a $|1p1h>$ final state $|P_F P_I>$ using eq. 3.3:

$$S_{fi} = \theta(P_{Fermi} - |\vec{P}_I|)\theta(|\vec{P}_F| - P_{Fermi}) \Big[J^{(1)}(P_I,P_F) - \sum_{p,\sigma,\tau} \theta(P_{Fermi} - |\vec{p}|) J^{(2)}(P_I,p,p,P_F) \Big] \tag{3.12}$$

where to compute $J^{(1)}, J^{(2)}$ the expressions 3.10, 3.11 are used. Once the transition matrix element is known, the cross-section is computed as in the previous sections.

D. The actual calculation

For the 2-body part the term in 3.12 corresponds to the sum of the two diagrams in figure 5.

For these diagram, we have to perform the summation on the loop:

1. *Diagram 5 (lower part)*

The momentum of the pion in the loop can be computed from the 4-momentum conserving $\delta^{(4)}$ function:

$$P_I + q = k_\pi + p = P_F \tag{3.13}$$

Note that as $|\vec{P}_F| > P_{Fermi}$ k_π cannot be 0. Besides the integration on $\vec{p}$ we have for the isospin part:

$$\sum_\tau \chi_F^\dagger \hat{O}_{\pi(-)} \chi_\tau \chi_\tau^\dagger \hat{O}_{\pi(+)} \chi_I = \chi_F^\dagger \hat{O}_{\pi(-)} \mathbf{1}_{2\times 2} \hat{O}_{\pi(+)} \chi_I \tag{3.14}$$

And for the spin sum:

$$\sum_\sigma \overline{u}_F \hat{J}^{\pi(-)} u_\sigma \overline{u}_\sigma \hat{J}_\mu^{\pi(+)} u_I = \overline{u}_F \hat{J}^{\pi(-)} (\not{p} + M) \hat{J}_\mu^{\pi(+)} u_I \tag{3.15}$$

---

[7]These operators can be found in Yasuo's notes, in eqs. 7.31 to 7.37, where one has to look at the $()_{ex}$ term and remove the factors $\frac{i}{(2\pi)^3}\frac{1}{\mu^2}$ and the $\delta^{(3)}$ function that we have already included in the formalism. The operators in between (2) and (2′) refer to the pion absorption ($J^{(-)}$) and the ones in between (1) and (1′) to the pion production ($J_\mu^{(+)}$). Note also that the g (strong) coupling constant in Yasuo's notes is not the g we have introduced here (either EM or weak coupling constant).

2. Diagram 5 (upper part)

This contribution is obtained from the previous one by interchanging $(1,1')$ by $(2,2')$. In this case,

$$P_I = k_\pi + p = P_F + q \tag{3.16}$$

and $|\vec{P}_I| < P_{Fermi}$, then $k_\pi = 0$ is an allowed value. The minimum (maximum in absolute value) of k_π^2 will be around $-2P_{Fermi}^2$ (-3 to -4 m_π^2). For the isospin part:

$$\sum_\tau \chi_F^\dagger \hat{\mathcal{O}}_{\pi(+)} \chi_\tau \chi_\tau^\dagger \hat{\mathcal{O}}_{\pi(-)} \chi_I = \chi_F^\dagger \hat{\mathcal{O}}_{\pi(+)} \mathbf{1}_{2\times 2} \hat{\mathcal{O}}_{\pi(-)} \chi_I \tag{3.17}$$

The spin sum:

$$\sum_\sigma \overline{u}_F \hat{J}_\mu^{\pi(+)} u_\sigma \overline{u}_\sigma \hat{J}^{\pi(-)} u_I = \overline{u}_F \hat{J}_\mu^{\pi(+)} (\not{p} + M) \hat{J}^{\pi(-)} u_I \tag{3.18}$$

In the nuclear medium, as the pion 4-momentum verifies either 3.13 or 3.16, we have $k_\pi^\mu k_{\pi\mu} < m_\pi^2$ (the pion is virtual). Due to the pion propagator in eq. 3.11, the integration in p is dominated for the maximum value of k_π^2, particularly for the $k_\pi^2 > 0$ case if these range of values is allowed as in 3.16. Soft pion limit expressions should be accurate at the 80% (or better) level for $T_F < 100$ MeV [23].

E. The summation on p, $SU(3)$ indices a

1. Isospin part

Using the convention for the $SU(3)$ generators λ^a matrices as in [20] we have the following equations useful when suming in τ, considering the isospin $SU(2)$ substructure of the $SU(3)$ algebra ($\tau_{(3,\pm)} \equiv \lambda^{(3,\pm)}$):

$$\sum_\tau \chi_\tau \chi_\tau^\dagger = \mathbf{1}_{2\times 2} \tag{3.19}$$

$$\sum_{a=1,2,3} \lambda^a \mathbf{1}_{2\times 2} \lambda_a = 3 \times \mathbf{1}_{2\times 2} \tag{3.20}$$

$$\sum_{a=1,2,3} \lambda^a \mathbf{1}_{2\times 2} \left[\lambda^a, \lambda^{(3,\pm)}\right]_- = 4 \times \lambda^{(3,\pm)} \tag{3.21}$$

$$\sum_{a=1,2,3} \left[\lambda^a, \lambda^{(3,\pm)}\right]_- \mathbf{1}_{2\times 2} \lambda^a = -4 \times \lambda^{(3,\pm)} \tag{3.22}$$

$$\sum_{a=1,2,3} \left[\lambda^a, \lambda^{(3,\pm)}\right]_+ \mathbf{1}_{2\times 2} \lambda^a = 2 \times \lambda^{(3,\pm)} \tag{3.23}$$

$$\sum_{a=1,2,3} \lambda^a \mathbf{1}_{2\times 2} \left[\lambda^a, \lambda^{(3,\pm)}\right]_+ = 2 \times \lambda^{(3,\pm)} \tag{3.24}$$

$$\sum_{a=1,2,3} \left[\lambda^a, \lambda^8\right]_+ \mathbf{1}_{2\times 2} \lambda^a = 2\sqrt{3} \times \mathbf{1}_{2\times 2} \tag{3.25}$$

$$\sum_{a=1,2,3} \lambda^a \mathbf{1}_{2\times 2} \left[\lambda^a, \lambda^8\right]_+ = 2\sqrt{3} \times \mathbf{1}_{2\times 2} \tag{3.26}$$

$$\left[\lambda^a, \lambda^8\right]_- = 0 \tag{3.27}$$

where $\lambda^\pm = \lambda^1 \pm i\lambda^2$ and

$$\mathbf{1}_{2\times 2} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.28}$$

2. Integration on $d\vec{p}$

The discrete summation on $\vec{p}$ is converted into continuous integration with the usual phase-space factor:

$$\sum_{\vec{p}} \to V \int \frac{d^3\vec{p}}{(2\pi)^3}$$

To perform the $\vec{p}$ integration one has to consider the dependence of the pion propagator $\Delta_\pi = \frac{i}{k_\mu k^\mu - m_\pi^2}$ on $k^\mu = (P_a - p)^\mu$ where $a = I, F$. The angular integration can be done analitycally, where we have to consider separately if the integrand has an explicit dependence on the direction of $\vec{p}$ or not. We choose the $z-$axis along $\hat{P}_a$, and denote $\mu = cos[\vec{p}, \vec{P}_a]$, then:

1. Terms without dependence on $\hat{p}$:

$$\int d\hat{p}\Delta_\pi = 2\pi \int_{-1}^{1} \frac{d\mu}{2M^2 - 2EE_a + 2|\vec{p}||\vec{P}_a|\mu - m_\pi^2} = \frac{\pi}{|\vec{p}||\vec{P}_a|} \log\left[\frac{2M^2 - 2EE_a + 2|\vec{p}||\vec{P}_a| - m_\pi^2}{2M^2 - 2EE_a - 2|\vec{p}||\vec{P}_a| - m_\pi^2}\right] \tag{3.29}$$

2. $\vec{p}$ terms:

$$\int d\hat{p}\vec{p}\Delta_\pi = 2\pi \int_{-1}^{1} \frac{\mu d\mu}{2M^2 - 2EE_a + 2|\vec{p}||\vec{P}_a|\mu - m_\pi^2} |\vec{p}|\hat{P}_a = \hat{P}_a \frac{2\pi}{|\vec{P}_a|}\left[1 + \frac{2EE_a + m_\pi^2 - 2M^2}{4|\vec{p}||\vec{P}_a|} \log\left[\frac{2M^2 - 2EE_a + 2|\vec{p}||\vec{P}_a| - m_\pi^2}{2M^2 - 2EE_a - 2|\vec{p}||\vec{P}_a| - m_\pi^2}\right]\right] \tag{3.30}$$

The remaining integration on $d|\vec{p}|$ can be performed numerically between 0 and P_{Fermi} including all dependence on E inside the integrand.

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